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ON THE MEAN VALUES OF AN ENTIRE FUNCTION REPRESENTED BY A DIRICHLET SERIES

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ABSTRACT
In this paper, we obtain some results for the mean value of an entire Dirichlet series.

THEOREM 1. $(i)\, For\, 0 < k < \infty, \delta > 1$

$$\genfrac{}{}{0pt}{}{\rho_*}{\lambda_*} \le \lim_{\sigma\to\infty} \genfrac{}{}{0pt}{}{\sup}{\inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \genfrac{}{}{0pt}{}{\rho}{\lambda} \qquad \text{(a)}$$

Under the additional condition on $\{\lambda_n\}$,

$$0 \le \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = D < \infty, \qquad \text{(A)}$$

(a) Becomes

$$\lim_{\sigma\to\infty} \genfrac{}{}{0pt}{}{\sup}{\inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} = \genfrac{}{}{0pt}{}{\rho}{\lambda} = \genfrac{}{}{0pt}{}{\rho_*}{\lambda_*} \qquad \text{(b)}$$

(ii) For $0 < k < \infty, \delta > 0$

$$\lim_{\sigma\to\infty} \genfrac{}{}{0pt}{}{\sup}{\inf} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \genfrac{}{}{0pt}{}{\rho}{\lambda} \qquad \text{(c)}$$

In fact for the truth of 'lim sup' part of (b) the following condition on $\{\lambda_n\}$ is sufficient.

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0. \qquad \text{(A')}$$

THEOREM 2. $(i)\, For\, \delta > 0,\ 0 < k < \infty,$

$$\lim_{\sigma\to\infty} \genfrac{}{}{0pt}{}{\sup}{\inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \genfrac{}{}{0pt}{}{T}{t}, (0 < \rho < \infty). \qquad \text{(d)}$$

$(ii)\, For\, \delta \ge 1,\ 0 < k < \infty$ and under the additional condition (A)

$$\genfrac{}{}{0pt}{}{T_*}{t_*} < \lim_{\sigma\to\infty} \genfrac{}{}{0pt}{}{\sup}{\inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \genfrac{}{}{0pt}{}{T}{t} < \genfrac{}{}{0pt}{}{T_*}{t_*} \genfrac{}{}{0pt}{}{e^{\rho D}}{e^{\rho D}} \qquad \text{(e)}$$

In particular case, if D = 0,

$$\lim_{\sigma\to\infty} \genfrac{}{}{0pt}{}{\sup}{\inf} \log \frac{N_{\delta,k}(\sigma)}{e^{\rho\sigma}} = \genfrac{}{}{0pt}{}{T_*}{t_*} = \genfrac{}{}{0pt}{}{T}{t} \qquad \text{(f)}$$

KEYWORDS: - Generalized order ρ
Generalized lower order λ

INTRODUCTION: In the usual notation,

$$f(s) = \sum_{1}^{\infty} a_n e^{s\lambda_n}, (s = \sigma + it), 0 < \lambda_n < \lambda_{n+1} \quad (n \ge 1) \lim_{n\to\infty} \lambda_n = \infty,$$

Is an entire function in the sense that the Dirichlet series representing it, is absolutely convergent for all finite s and possesses two generally different pairs of orders:

$$\lim_{n\to\infty} \genfrac{}{}{0pt}{}{\sup}{\inf} \frac{\log\log M(\sigma)}{\sigma} = \genfrac{}{}{0pt}{}{\rho}{\lambda};$$

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\sigma} = \frac{\rho_*}{\lambda_*};$$

Where $0 \le \lambda, \rho \le \infty, 0 \le \lambda_*, \rho_* \le \infty, and\ M(\sigma), \mu(\sigma)$ their usual meanings, viz.

$$M(\sigma) = \underset{-\infty < t < \infty}{l.u.b.} \left|f(\sigma+it)\right|, \quad \mu(\sigma) = \max_{n\ge 1}\left|a_n e^{(\sigma+it)\lambda_n}\right|$$

The type T, t associated with ρ and type T*, t_* associate with ρ_* are defined in the usual way as follow:

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\rho\sigma}} = \frac{T}{t}, \quad (0 < \rho < \infty)$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma)}{e^{\rho_*\sigma}} = \frac{T_*}{t_*}, \quad (0 < \rho_* < \infty).$$

The mean values of t (s) are defined as follows:

$$\{I_\delta(\sigma)\}^\delta = A_\delta(\sigma) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|f(\sigma+it)\right|^\delta dt, \quad 0 < \delta < \infty, \qquad (1.1)$$

$$N_{\delta,k}(\sigma) = \frac{1}{e^{k\sigma}}\int_{-\infty}^{\sigma} I_\delta(x) e^{kx} dx$$

$$= \lim_{T\to\infty} \frac{1}{2Te^{k\sigma}} \int_{-\infty}^{\sigma}\int_{-T}^{T} \left|f(x+it)\right|^\delta e^{kx} dx\, dt \;, \quad \begin{matrix}0<\delta<\infty\\ 0<k<\infty\end{matrix} \qquad (1.2)$$

Clearly $\rho_* \le \rho$ and $\lambda_* \le \lambda$. There are entire Dirichlet series for which $\rho_* < \rho, \lambda_* < \lambda$ (sec[9], *Satz* 4). So, we have generally to distinguish between the two orders of an entire Dirichlet series and its types associated with these orders.

THEOREM 1. (i) *For* $0 < k < \infty,\ \delta \ge 1$

$$\frac{\rho_*}{\lambda_*} \le \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \frac{\rho}{\lambda} \qquad (2.1)$$

Under the additional condition on $\{\lambda_n\}$,

$$0 \le \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log n}{\lambda_n} = D < \infty, \qquad (A)$$

(2.1) becomes

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} = \frac{\rho}{\lambda} = \frac{\rho_*}{\lambda_*} \qquad (2.2)$$

(ii) For $0 < k < \infty, \delta > 0$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \frac{\rho}{\lambda} \qquad (2.3)$$

In fact for the truth of 'lim sup' part of (2.2) the following condition on $\{\lambda_n\}$ is sufficient.

$$\lim_{\sigma\to\infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0. \qquad (A')$$

Proof. For fixed σ,

$$f(\sigma+it) = \sum_{1}^{\infty} (a_n e^{\lambda_n\sigma}) e^{i\lambda_n t}, \quad (-\infty < t < \infty)$$

is an absolutely and uniformly convergent function of *t* and hence ([2], *p*.6) a function of *t* which is uniformly almost periodic (briefly *u.a.p.)* $\left|f(\sigma+it)\right|^{\delta}, \delta>0$ is also a function of *t* which is *u.a.p.,* as shown by familiar considerations (*e.g.* as in [2] p.3) involving the following well known inequalities for *a >0, b >0.*

$(a+b)^{\delta} \le a^{\delta}+b^{\delta} if\ 0<\delta<1, a^{\delta}-b^{\delta} \le \delta_a^{\delta-1}(a-b),\ \ if\ \delta \ge 1, a \ge b.$

By the result ([2], p. 12) the mean value of $\left|f(\sigma+it)\right|^{\delta}, \delta>0,$ defined by $A_{\delta}(\sigma)$ exists.

For $\delta>0$ it is obvious that

$$I_{\delta}(\sigma) \le M(\sigma).$$

This, with (1.2) will give us

$$N_{\delta,k}(\sigma) \le \frac{M(\sigma)}{k}. \qquad (2.4)$$

From which it follows that

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \begin{matrix}\rho\\ \lambda\end{matrix},\ 0<k<\infty,\ \delta>0. \qquad (2.5)$$

This formula gives us

$$\mu(\sigma) \le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left|f(\sigma+it)\right|dt = I_1(\sigma)$$

If δ >1, we also get by Holder integral inequality $\mu(\sigma) \le \lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T}\left|f(\sigma+it)\right|^{\delta}dt\right]^{\frac{1}{\delta}}\left[\frac{1}{2T}\int_{-T}^{T}dt\right]^{\frac{1}{\delta'}}$

where $\frac{1}{\delta}+\frac{1}{\delta'}=1$. Hence

$$\mu(\sigma) \le I_{\delta}(\sigma)\ \ for\ \delta \ge 1.$$

From (1.2), we have for *h >0,*

$$N_{\delta,k}(\sigma+h) \ge \frac{\mu(\sigma)}{k}(1-e^{-kh}) \qquad (2.6)$$

This leads to

$$\frac{\log\log N_{\delta,k}(\sigma+h)}{(\sigma+h)} \ge \frac{\log\log\mu(\sigma)}{(\sigma+h)}+o(1)$$

Proceedings to limits, we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \ge \begin{matrix}\rho_*\\ \lambda_*\end{matrix} \qquad (2.7)$$

Combining (2.5) and (2.7), we get

$$\begin{matrix}\rho_*\\ \lambda_*\end{matrix} < \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log N_{\delta,k}(\sigma)}{\sigma} \le \begin{matrix}\rho\\ \lambda\end{matrix}$$

To prove (2.2), we use the known result ([10], p. 68) that, under the condition (A),

$$M(\sigma) < K\,\mu(\sigma+D+\varepsilon)$$

where ε is an arbitrary small positive number, K is a constant depending on D and ε. This gives $\rho \le \rho_*\ and\ \lambda \le \lambda_*$ but $\rho_* < \rho\ and\ \lambda_* < \lambda$ always. Thus, (2.2) proved.

It is known that under the condition (A')

$$\rho = \lim_{n\to\infty}\sup\frac{\lambda_n\log\lambda_n}{\log\left|a_n\right|^{-1'}}, [1].$$

Further, from the result of Reddy [8] we conclude that

$$\rho_* = \lim_{n\to\infty}\sup\frac{\lambda_n\log\lambda}{\log\left|a_n\right|^{-1'}}.$$

Thus, we have completed the proof of the theorem

THEOREM 2. $(i)\, For\, \delta > 0, 0 < k < \infty,$

$$\lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le {}^{T}_{t}, \; (0 < \rho < \infty). \tag{3.1}$$

$(ii)\, For\, \delta \ge 1, 0 < k < \infty$ and under the condition (A)

$${}^{T_*}_{t_*} \le \lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le {}^{T}_{t} \le {}^{T_*}_{t_*} \; {}^{e^{\rho D}}_{e^{\rho D}} \tag{3.2}$$

In particular case, if D = 0,

$$\lim_{\sigma\to\infty} {}^{\sup}_{\inf} \log \frac{N_{\delta,k}(\sigma)}{e^{\rho\sigma}} = {}^{T_*}_{t_*} = {}^{T}_{t} \tag{3.3}$$

Proof. From (2.4), we get

$$\lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log M(\sigma)}{e^{\rho\sigma}}.$$

From which (3.1) follows.

To prove (3.2), we use (2.4), (2.6) and the known result $M(\sigma) < K\,\mu(\sigma + D + \varepsilon)$[10], where ε is an arbitrary small positive number and K is constant depending on D and ε . We have

$$\lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log \mu(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log M(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log \mu(\sigma + D + \varepsilon)}{e^{\rho\sigma}}$$

And

$$\lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log \mu(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log N_{\delta,k}(\sigma)}{e^{\rho\sigma}} \le \lim_{\sigma\to\infty} {}^{\sup}_{\inf} \frac{\log M(\sigma)}{e^{\rho\sigma}}.$$

Combining these two, we get desired conclusion (3.2). The particular case (3.3) is obvious.

Conclusion: **.** Our theorem includes the results of Jain [5], which in turn includes the theorem of Juneja [6] and also a theorem of Gupta [3]. The method of proofs of our results is different from that of Jain. Jain has used in his proof the following result of Kamthan [7]

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