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MATHEMATICAL MODELING OF THE HYDRODYNAMIC STABILITY PROBLEM BY THE SPECTRAL-GRID METHOD

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ABSTRACT

In the article, using the method of small perturbations, mathematical models of hydrodynamic stability for single-phase flows are obtained. The spectral-grid method is used to approximate the stability equations. It combines the high accuracy of the spectral method of nonuniform grids and allows one to determine all the eigenvalues of the problem under consideration at once. In the spectral-grid method, the interval of integration with respect to the spatial variable is divided into a grid; in the grid elements, the approximate solution is approximated using a linear combination of a different number of series in Chebyshev polynomials of the first kind. Among orthogonal polynomials, only Chebyshev polynomials have the minimax property, i.e. for these polynomials, the maximum deviation from the desired solution is minimal. In addition, there are convenient recurrence formulas for the computational application of Chebyshev polynomials. Using these formulas, you can easily calculate the values of polynomials and their derivatives of the desired order.

KEYWORDS: hydrodynamic stability, Reynolds number, wavenumber, integration interval, grid, approximation, Chebyshev polynomials of the first kind, eigenvalues, single-phase, main flow, laminar flow.

INTRODUCTION

In papers [1-6] the study of methods for finding the eigenvalue which is the coefficients of the Orr-Sommerfeld equation. In [7], a study of one effective method for solving the Orr-Sommerfeld equation. And in work [8-11] mathematical models of the problem of hydrodynamic stability of single-phase flows were created.

The spectral-grid method (SGM) is a new effective mathematical apparatus for numerical modeling of the hydrodynamic stability problem. It combines the high accuracy of spectral methods with the economy of the nonuniform mesh method and allows one to determine all the eigenvalues of the problem at once. In this article, an effective research method for hydrodynamic stability has been found.

In this work, to overcome the above difficulties, the spectral-grid method is used [12-13]. Depending on the type of initial data or the expected form of the solution, a grid is introduced in the integration interval. At the internal nodes of the grid, the requirement is imposed on the continuity of the solution and its derivatives up to order $m-1$, where m the order of the highest derivative of the differential equation is. At the boundary grid nodes, the corresponding boundary conditions for the problem under consideration are set. An approximate solution on grid elements is represented in the form of finite series in Chebyshev polynomials of the first kind. The resulting system of equations with the help of linear non-degenerate transformations is reduced to two autonomous systems: a linear system of algebraic equations and a system (in the general case, nonlinear) of ordinary differential equations. To solve the first system, standard methods are used, and to solve the second, an explicit algorithm developed by.

Therefore, the use of the spectral-grid method makes it possible, firstly, to distribute the Chebyshev polynomials over the elements, taking into account the behavior of the solution gradient and, secondly, to lead to a significant decrease in the order of the matrices in the arising algebraic system. In this method, for a given number of grid elements N, to achieve the required accuracy of calculations, it is necessary to correctly position the grid nodes and select the number of polynomials p_j on the grid elements. These questions are closely related, because by bringing the grid nodes closer together, one can reduce the number of polynomials on the elements and vice versa. In practical calculations, it is more convenient to choose a uniform mesh, specifying a different number of polynomials p_j on each mesh element. Then the number of required polynomials depends on the relative value of the gradients of the solution on a particular element [14-16].

Mathematical models characterizing the viscous motion of a viscous incompressible fluid are described by the Navier-Stokes equation [6-8]:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\vartheta\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$
$$\frac{\partial \vartheta}{\partial t}+u\frac{\partial \vartheta}{\partial x}+\vartheta\frac{\partial \vartheta}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 \vartheta}{\partial x^2}+\frac{\partial^2 \vartheta}{\partial y^2}\right), \tag{1}$$
$$\frac{\partial u}{\partial x}+\frac{\partial \vartheta}{\partial y}=0,$$

where u,ϑ - longitudinal and transverse components of velocity, p - pressure, $Re=\rho UL/\mu$ - Reynolds numbers, ρ - density, μ - fluid viscosity, U and L - characteristic scales of velocity and length, respectively.

MAIN PART

To study the stability of the solution to system (1), we represent, as usual, in the form of a superposition of the main laminar flow $U(y)$ and a small perturbation:

$$u(x,y,t)=U(y)+\tilde{u}(x,y,t),$$
$$\vartheta(x,y,t)=\tilde{\vartheta}(x,y,t), \tag{2}$$
$$p(x,y,t)=P(x,y)+\tilde{p}(x,y,t).$$

We write system (1) taking into account (2) and, leaving in the obtained equations only terms of the first order of smallness in perturbations, we have

$$\frac{\partial \tilde{u}}{\partial t}+U\frac{\partial \tilde{u}}{\partial x}+\tilde{\vartheta}\frac{dU}{dy}+\frac{\partial P}{\partial x}+\frac{\partial \tilde{p}}{\partial x}=\frac{1}{Re}\left(\frac{d^2U}{dy^2}+\Delta\tilde{u}\right),$$
$$\frac{\partial \tilde{\vartheta}}{\partial t}+U\frac{\partial \tilde{\vartheta}}{\partial x}+\frac{\partial P}{\partial y}+\frac{\partial \tilde{p}}{\partial y}=\frac{1}{Re}\Delta\tilde{\vartheta}, \tag{3}$$
$$\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{\vartheta}}{\partial y}=0, \quad \Delta=\partial^2/\partial x^2+\partial^2/\partial y^2.$$

Considering that the main flow itself satisfies the Navier-Stokes equations, i.e.

$$\frac{\partial P}{\partial x}=\frac{1}{Re}\frac{d^2U}{dy^2}, \quad \frac{\partial P}{\partial y}=0,$$

then system (3) takes the form

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \tilde{\vartheta}\frac{dU}{dy} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}\Delta \tilde{u},$$

$$\frac{\partial \tilde{\vartheta}}{\partial t} + U\frac{\partial \tilde{\vartheta}}{\partial x} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re}\Delta \tilde{\vartheta}, \qquad (4)$$

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{\vartheta}}{\partial y} = 0. \qquad (5)$$

We introduce the stream function for the disturbing motion in the form

$$\Psi(x, y, t) = \psi(y) \cdot e^{i(kx-\omega t)}, \qquad (6)$$

where $\psi(y) = \psi_r + i\psi_i$ is the complex amplitude of perturbations, k is a real value associated with the length of the ℓ -wave of perturbation by the relation $\ell = 2\pi/k$. Value ω is complex, $\omega = \omega_r + i\omega_i$, where ω_r is the angular frequency of an individual vibration, and ω_i is the rise factor, i.e. a value that makes it possible to judge whether the oscillation increases or dies. If $\omega_i < 0$, then the oscillation decays and the laminar flow is stable, but if $\omega_i > 0$, then instability takes place. In addition to the quantities k and ω, it is advisable to also introduce their ratio $\lambda = \omega / k = \lambda_r + i\lambda_i$. The quantity λ_r is the propagation velocity of the waves in direction x (phase velocity), and λ_i is again a quantity that allows one to judge about the decay or increase of the oscillation.

The amplitude $\psi(y)$ of the disturbing motion is taken to depend only on variable y, because the main flow also depends only on y. For stream function (6) we have

$$\tilde{u} = \frac{\partial \Psi}{\partial y} = \psi'(y)e^{i(kx-\omega t)}, \quad \tilde{\vartheta} = -\frac{\partial \Psi}{\partial x} = -ik\psi(y)e^{i(kx-\omega t)},$$

thus, the continuity equation (5) is integrated, and from system (4) we obtain the eigenvalue problem for the Orr - Sommerfeld equation [1-6]:

$$\frac{1}{\text{ikRe}}\text{D}^2\psi - \left((U(\eta) - \lambda)D + \frac{d^2U}{d\eta^2}\right)\psi = 0, \qquad (7)$$

$$\eta_0 < \eta < \eta_l \ ,$$

$$\psi(\eta_0) = \frac{d\psi}{d\eta}(\eta_0) = 0, \quad \psi(\eta_l) = \frac{d\psi}{d\eta}(\eta_l) = 0 \qquad (8)$$

with uniform boundary conditions, which means impermeability and adhesion requirements. Here $D = \frac{d^2}{d\eta^2} - k^2$ is the differential operator, $U(\eta)$ is the velocity profile of the main flow, η is the coordinate directed across the main flow, k is the wave number, Re is the Reynolds number, $\psi(\eta)$ are the amplitudes of the stream function for perturbations, $\lambda = \lambda_r + i\lambda_i$ are the eigenvalues of the problem, where λ_r is the phase the speed of the wave disturbance, λ_i is the growth factor. If $\lambda_i > 0$, then the flow is unstable, if $\lambda_i < 0$, then it is stable. If $\lambda_i = 0$, then the oscillations are neutral stable.

SGM to simulate the equation of stability for single-phase hydrodynamic systems described by the eigenvalue problem (7) - (8). We divide the integration interval $[\eta_0, \eta_l]$ into a grid and get N different elements:

$$[\eta_0, \eta_1], [\eta_1, \eta_2], ..., [\eta_j, \eta_{j+1}], ..., [\eta_{N-1}, \eta_N].$$

Differential equation (7) on each of these elements takes the form

$$D^2\psi_j - ikRe\left[\left(U_j(\eta) - \lambda\right)D - U_j''(\eta)\right]\psi_j = 0, \quad j = 1, 2, ..., N \qquad (9)$$

Boundary conditions (8) are written at points η_0 and η_N :

$$\psi_1(\eta_0) = \frac{d\psi_1}{d\eta}(\eta_0) = 0, \qquad \psi_N(\eta_N) = \frac{d\psi_N}{d\eta}(\eta_N) = 0. \qquad (10)$$

At the internal nodes of the grid, we require the continuity of the solution to equation (9) and its derivatives up to the third order:

$$\psi_j^{(t)}(\eta_j) = \psi_{j+1}^{(t)}(\eta_j), \quad t = 0,1,2,3; \quad j = 1,2,\ldots, N-1 \qquad (11)$$

where t indicates the order of the derivative.

To represent the solution of equations (9) - (11) in the form of a series in Chebyshev polynomials of the first kind, each element $[\eta_j, \eta_{j+1}]$ is mapped to the interval $[-1,1]$. After this transformation, equations (9) take the form

$$L_j\psi_j = \left\{D_j^2 - ik_j \operatorname{Re}_j\left[\left(U_j(o') - \lambda\right)D_j - U_j''(o')\right]\right\}\psi_j = 0 \qquad (12)$$

$$j = 1,2,\ldots, N,$$

where

$$D_j = \frac{d^2}{dy_j^2} - k_j^2, \quad k_j = \frac{l_j}{2}k, \quad Re_j = \frac{l_j}{2}Re.$$

From conditions (10) - (11) we have

$$\psi_1(-1) = 0, \frac{d\psi_1}{dy}(-1) = 0,$$

$$l_j^{-t}\psi_j^{(t)}(+1) = l_{j+1}^{-t}\psi_{j+1}^{(t)}(-1),$$

$$t = 0,1,2,3; \quad j = 1,2,\ldots, N-1, \qquad (13)$$

$$\psi_N(+1) = 0, \quad \frac{d\psi_N}{dy}(+1) = 0$$

$l_j = \eta_j - \eta_{j-1}$ indicates the length of the j th mesh element.

We seek an approximate solution to problem (12) - (13) on each of the grid elements in the form

$$\psi_j(y) = \sum_{n=0}^{p_j} a_n^{(j)} T_n(y),$$

$$U_j\left(y_e^j\right) = \sum_{n=0}^{p_j} b_n^{(j)} T_n\left(y_e^j\right), \qquad (14)$$

$$y_e^j = \cos\left(\pi \cdot l/p_j\right), \quad l = 0,1,2,\ldots, p_j; \quad j = 1,2,\ldots, N,$$

where $T_n(y)$ are the Chebyshev polynomials of the first kind, y_l^j are their nodes, and p_j is the number of polynomials used to approximate the solution at the j th element.

Substituting series (14) into equation (12), we require that the left side of (12) on each of the grid elements be orthogonal to the first $p_j - 4$ Chebyshev polynomials:

$$\left(L_j\psi_j, T_n\right) = 0, \quad n = 0,1,\ldots, p_j - 4, \quad j = 1,2,\ldots, N, \qquad (15)$$

where

$(f,g) = \int_{-1}^{+1} f(x)g(x)(1-x^2)^{-1/2}dx$ – dot product on segment $[-1,1]$. In addition, we also require that the series in Chebyshev polynomials (14) exactly satisfy the boundary and continuity conditions (13). Taking into account the following properties of Chebyshev polynomials $T_n(\pm 1) = (\pm 1)^n$ and $T_n^{'}(\pm 1) = (\pm 1)^{n-2} n^2$, these conditions are written in the form:

$$\sum_{n=0}^{p_1} (-1)^n a_n^0 = 0, \quad \sum_{n=0}^{p_1} (-1)^{n-1} n^2 a_n^{(1)} = 0, \quad \sum_{n=0}^{p_j} a_n^{(j)} = \sum_{n=0}^{p_{j+1}} (-1)^n a_n^{(j+1)},$$

$$\frac{1}{l_j} \sum_{n=0}^{p_j} n^2 a_n^{(j)} = \frac{1}{l_{j+1}} \sum_{n=0}^{p_{j+1}} (-1)^{n-1} n^2 a_n^{(j+1)},$$

$$\frac{1}{l_j^2} \sum_{n=0}^{p_j} a_n^{(j)} T_n^{''}(+1) = \frac{1}{l_{j+1}^2} \sum_{n=0}^{p_j} a_n^{(j+1)} T_n^{''}(-1),$$

$$\frac{1}{l_j^3} \sum_{n=0}^{p_j} a_n^{(j)} T_n^{'''}(+1) = \frac{1}{l_{j+1}^3} \sum_{n=0}^{p_{j+1}} a_n^{(j+1)} T_n^{'''}(-1), \quad \mathrm{j} = 1,2,3,\ldots, \mathrm{N}-1$$

$$\sum_{n=0}^{p_N} a_n^{(N)} = 0, \quad \sum_{n=0}^{p_N} n^2 a_n^{(N)} = 0. \qquad (16)$$

Thus, to determine $\overline{m} = N(p_j + 1)$ unknowns $a_n^{(j)} n = 0,1,2,\ldots, p_j$; $j = 1,2,3\ldots, N$, we have $\overline{m} = N(p_j + 1)$ equations. These equations are $N(p_j - 3)$ orthogonality equations (15), $4(N-1)$ continuity conditions, and four boundary conditions from (16). The resulting system can be conveniently written in matrix form:

$$(A - \lambda B)X = 0, \qquad (17)$$

where complex matrices A and B have a beam-diagonal structure of a special type, and vector X contains coefficients $a_n^{(j)}$ in expansion (14), i.e.

$$x^T = (a_0^0, a_1^0, \ldots, a_{P_1}^0, a_0^0, a_1^0, \ldots, a_{P_2}^0, \ldots, a_0^{(N)}, a_1^{(N)}, \ldots, a_{P_N}^{(N)}).$$

A characteristic feature of system (17) is that matrix B is degenerate (since conditions (16) do not depend on λ) and contains $4N$ zero rows, where N is the number of grid elements.

RESULTS AND DISCUSSION

The construction of an algebraic transformation for matrix stability equations (17) is presented in the fourth section. This system, using a nondegenerate linear transformation Q, is reduced to the form

$$(AQ - \lambda BQ)(Q^{-1}x) = 0. \qquad (18)$$

After applying transformation Q, the number of rows and columns of complex matrices A and B is reduced by $4N$, where N is the number of elements. The ratio of the total number of equations $\overline{m}$ to the number of the remaining $\overline{m} = 4N$ equations is

$$q = \frac{\overline{m}}{\overline{m} - 4N} \qquad (19)$$

Thus, as a result of dividing the integration interval into elements, the dimension of each complex matrices (real and imaginary parts) A and B in the original algebraic system decreases by q^2 times. The reduction in dimension is especially noticeable with a small number of polynomials at each of the elements. Indeed, the number of polynomials on the jth element is $p_j + 1$ $(j = 1,2,3,\ldots, N)$. Then the total number of polynomials and, accordingly, algebraic equations is $\overline{m} = \sum_{j=1}^{N} (p_j + 1)$. Note that p_j should not be less than the order of the highest derivative of the differential equation, i.e. $p_j \geq 4$. For $p_j = 4$, for example, for all

j, $\overline{m} = 5N$, $q = 5$. This means that the number of equations in the system is reduced by 5 times, and the dimensions of each complex matrices A and B by 25 times.

From the remaining equations, an algebraic system of significantly lower dimension is obtained:

$$(\mathrm{T} - \lambda W)Y = 0, \tag{20}$$

$$Y = Q^{-1}\overline{x}, \qquad \overline{x} = (a_4^0, ..., a_{p_1}^0, a_4^0, ..., a_{p_2}^0, ..., a_4^{(N)}, ..., a_{p_N}^{(N)})$$

where W is generally a non-degenerate square matrix.

Multiplying (20) on the left by matrix W^{-1}, we obtain

$$(D - \lambda E)Y = 0, D = TW^{-1}. \tag{21}$$

The eigenvalues of system (21) can be found by standard methods. In this work, they are determined using the QR - algorithm.

To solve a system of the form (21), one step of the QR -algorithm requires $Z = (\frac{20}{3})\bar{n}^3$ arithmetic operations. Table 1 compares spectral method SM and SGM by the number of arithmetic operations Z.

Table 1.

$\overline{m}$	SM	SGM	
	Z	N	Z
5	6	1	6
10	1440	2	53
20	27306	4	426
30	117173	6	1440
40	311040	8	3414
50	648906	10	6666
60	1170773	12	27306
70	1916640	14	70986
80	2926507	16	146346
90	4240373	18	262026
100	5898240	20	426666

Efficiency SGM, given in table. 1. The most clearly illustrated in fig. 1, where Z denotes the number of arithmetic operations.

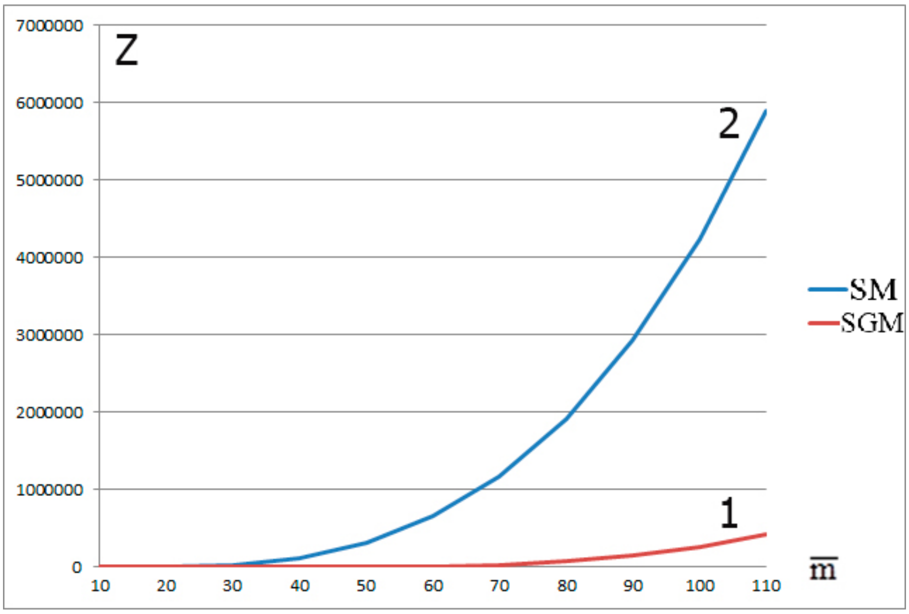

Fig. 1 Curve 1 - SGM, curve 2 - SM.

However, the high accuracy of the SGM is maintained.

CONCLUSIONS

The spectral-grid method and the spectral method are compared by the number of arithmetic ones when solving the standard eigenvalue problem with a complex matrix. It is shown that the spectral-grid method is economical and has high accuracy in solving the problem of hydrodynamic stability.

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