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ON THE INEQUALITIES CONCERNING POLYNOMIAL-EXPONENTIAL BOUNDS FOR INVERSE TRIGONOMETRIC FUNCTION

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Abstract

Recent research works on mathematical inequalities shows the importance bounds of polynomial-exponential type for various functions. In this article, we have provided polynomial-exponential type bounds $\left(1 \pm \frac{1}{4}x^2\right)e^{\alpha x^2}$ for inverse trigonometric function $\sin^{-1} x/x$ which refines the inequalities existed in the literature.

Keywords: Arcsine function; Polynomial-Exponential bounds; Inverse trigonometric functions.

MSC: 26D05; 26D07; 26D20; 33B10

Introduction:

The arcsine function is useful in navigation, engineering, and other sciences. Traditionally, it is used to determine the measure of an angle between the known two sides of right angled triangle. The graph of this function on domain $(0,1)$ is bounded between 1 and $\pi/2$, and many researchers have studied refined bounds of it. We depict some of them here, which are of our interest.

One of the complicated approximations to arcsine function was established by Edward Neuman [9] given by (1.1) in terms of inverse tangent function,

$$\left(\frac{\tanh^{-1}\omega^2}{\omega^2}\right)^2 \leq \frac{\sin^{-1}x}{x} \leq \left(\frac{\tanh^{-1}\omega^2}{\omega^2(1-\omega^4)}\right)^{\frac{1}{2}}, \text{where } \omega^2 = \frac{1-\sqrt{1-x^2}}{2} \qquad (1.1)$$

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Dhaigude & Thool [7] established a double inequality (1.2) with exponential bounds of polynomial kind,

$$\frac{1}{\left(1-\frac{2x^2}{3}\right)^{\eta_1}} < \frac{\sin^{-1}x}{x} < \frac{1}{\left(1-\frac{2x^2}{3}\right)^{\eta_2}} \qquad (1.2)$$

where $x \in (0,1)$, l is any real number in $(0,1)$, and $\eta_1 = 1/4$ and $\eta_2 = \ln\left(\frac{l}{\sin^{-1}l}\right)/\ln\left(1-\frac{2l^2}{3}\right)$ are the best possible constants s.t. (1.2) holds.

$$1+\frac{x^2}{6} < \frac{\sin^{-1}x}{x} < 1+\left(\frac{\pi-2}{2}\right)x^2, x \in (0,1) \qquad (1.3)$$

$$\frac{6}{6-x^2} < \frac{\sin^{-1}x}{x} < \frac{\pi}{\pi-(\pi-2)x^2}, x \in (0,1) \qquad (1.4)$$

Dhaigude & Bagul [6] have given simple efficient bounds in the double inequalities (1.3) and (1.4), where one can observe that (1.4) is the refinement of (1.3). The exponential type of bounds for arcsine function was corroborated by Bagul and Bagul et.al. in [2] and [3] respectively, the corresponding double inequality is given by (1.5) as follows

$$e^{x^2/6} < \frac{\sin^{-1}x}{x} < e^{\ln(\pi/2)x^2}, x \in (0,1) \qquad (1.5)$$

The double inequality (1.5) provides sharp lower and upper bounds than that of (1.3). Moreover, the lower bound in (1.4) is sharper than that of the lower bound in (1.5). One can more details in [2, 3, 7, 9] and references therein about the inequalities of arcsine function.

Recently, the polynomial-exponential type have been discussed for an exponential function by Chesneau [5]. Also, Bagul et.al. [4] have discussed the bounds of polynomial-exponential type for sinc and hyperbolic sinc functions. In [8], the authors have discussed the same bounds for tangent function. This shows that the researchers are quite interested in finding such type of bounds, which motivates us to write this article, by providing polynomial-exponential type bounds $\left(1 \pm \frac{1}{4}x^2\right)e^{\alpha x^2}$ for arcsine function. We will discuss them in next sections.

Two Theorems:

This section is dedicated for the two theorems, their statements are as follows:

Theorem 2.1

The double inequality (2.1) holds true for all real numbers in $(0,1)$,

$$\left(1-\frac{x^2}{4}\right)\ e^{\ln\left(\frac{8}{3\pi}\right)x^2} < \frac{x}{\sin^{-1}x} < \left(1-\frac{x^2}{4}\right)\ e^{x^2/12}, \tag{2.1}$$

where $\ln\left(\frac{8}{3\pi}\right)$ and $\frac{1}{12}$ are the best possible constants.

Theorem 2.2

The double inequality (2.2) holds true for all real numbers in $(0,1)$,

$$\left(1+\frac{x^2}{4}\right)\ e^{ln\left(\frac{8}{5\pi}\right)x^2} < \frac{x}{sin^{-1}x} < \left(1+\frac{x^2}{4}\right)\ e^{-(5/12)x^2}, \tag{2.2}$$

where $ln\left(\frac{8}{5\pi}\right)$ and $\frac{-5}{12}$ are the best possible constants.

Preliminaries & Lemmas:

In this section, we write some preliminary results and required lemmas that are required to prove theorems 2.1 and 2.2. First we state the celebrated lemma which is known as l'Hˆopital's rule of monotonicity ([1] p. 10)(see also [11, 12]).

Lemma 3.1 (l'Hˆopital's rule of monotonicity, [1,11,12])

Let α, β be two real valued functions which are continuous on $[a,b]$ and differentiable on (a,b), where $-\infty < a < b < \infty$ and $\beta'(x) \neq 0,\ \forall x \in (a,b)$. Let,

$$r_1(x) = \frac{\alpha(x) - \alpha(a)}{\beta(x) - \beta(a)}\ and\ r_2(x) = \frac{\alpha(x) - \alpha(b)}{\beta(x) - \beta(b)}.$$

Then

i) $r_1(x)$ and $r_2(x)$ are increasing on (a,b) if $\frac{\alpha'}{\beta'}$ is increasing on (a,b) and

ii) $r_1(x)$ and $r_2(x)$ are decreasing on (a,b) if $\frac{\alpha'}{\beta'}$ is decreasing on (a,b).

The strictness of the monotonicity of $r_1(x)$ and $r_2(x)$ depends on the strictness of monotonicity of $\frac{\alpha'}{\beta'}$.

Next lemma gives us, the power series expansion of arcsine function that can be seen in ([10], 1.645).

Lemma 3.2

If $x \in (0,1)$, then we have

$$\frac{sin^{-1}x}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} a_k x^{2k+1} \tag{3.1}$$

and

$$(sin^{-1} x)^2 = \sum_{k=0}^{\infty} b_k x^{2k+2} \qquad (3.2)$$

where $a_k = (k+1)b_k \ \ and \ \ b_k = \frac{2^{2k}(k!)^2}{(2k+1)!(k+1)}$.

Now, we provide four lemmas which are key ingredients to prove the main theorems of this article.

Lemma 3.3

If $k = 0, 1, 2, 3, \dots \dots$, then

$$2a_k - 17a_{k+1} + 40a_{k+2} - 16a_{k+3} + 2b_k + 14b_{k+1} - 48b_{k+2} + 32b_{k+3} > 0. \quad (3.3)$$

Lemma 3.4

If $x \in (0,1)$, then

$$x^2(x^2-4)^2 + 2(x^4 + 8x^2 - 16)(1-x^2)(sin^{-1} x)^2 < \frac{(2x^3 - x)(x^2-4)^2(sin^{-1} x)}{\sqrt{1-x^2}} \qquad (3.4)$$

Lemma 3.5

If $k = 0, 1, 2, 3, \dots \dots$, then

$$2a_k + 15a_{k+1} + 24a_{k+2} - 16a_{k+3} + 2b_k - 18b_{k+1} - 16b_{k+2} + 32b_{k+3} > 0 \quad (3.5)$$

Lemma 3.6

If $x \in (0,1)$, then

$$x^2(x^2+4)^2 + 2(x^4 - 8x^2 - 16)(1-x^2)(sin^{-1} x)^2 < \frac{(2x^3 - x)(x^2+4)^2(sin^{-1} x)}{\sqrt{1-x^2}} \qquad (3.6)$$

4. Proof of some Lemmas & Theorems:

Proof of **Lemma 3.3:**

Denote the R.H.S. of (3.3) by $m(k)$. We have to prove $0 > -m(k) = M(k)$ (say). Utilising **Lemma 3.2** in $M(k)$, we get

$$M(k) = 2a_k - 17a_{k+1} + 40a_{k+2} - 16a_{k+3} + 2b_k + 14b_{k+1} - 48b_{k+2} + 32b_{k+3} = \frac{2^{2k} \times r(k)}{(2k+1)!\,(8k^7 + 140k^6 + 1022k^5 + 4025k^4 + 9212k^3 + 12215k^2 + 8658k + 2520)}$$

where

$$r(k) = -36k^7 - 600k^6 - 4011k^5 - 14312k^4 - 30182k^3 - 38227k^2 - 26980k - 8160$$

Observe that, $r(k)$ is negative for all $k = 0, 1, 2, \dots \dots$

Therefore, $M(k) < 0$ and hence $m(k) > 0, \forall\, k$. This proves inequality (3.3).

Proof of **Lemma 3.4:**
To prove inequality (3.4), first we rewrite it and denote new form by

$P(x) = x^2(x^2-4)^2 + A(x) + B(x) < 0.$

Now we prove that $P(x)$ is negative decreasing function, where

$$A(x) = \frac{(x-2x^3)(x^2-4)^2(sin^{-1}x)}{\sqrt{1-x^2}} \ and$$
$$B(x) = 2(x^4+8x^2-16)(1-x^2)(sin^{-1}x)^2$$

Now, by utilizing **Lemma 3.2** in $A(x) and\ B(x)$, we simply get the following

$$A(x) = -\frac{17}{15}x^6 - \frac{88}{3}x^4 + 16x^2 + \sum_{k=0}^{\infty} A_k\, x^{2k+8} \ and$$
$$B(x) = -\frac{83}{45}x^6 + \frac{56}{3}x^4 - 16x^2 + \sum_{k=0}^{\infty} B_k\, x^{2k+8}$$

Where

$$A_k = -2a_k + 17a_{k+1} - 40a_{k+2} + 16a_{k+3} \ and$$
$$B_k = -b_k - 7b_{k+1} + 24b_{k+2} - 16b_{k+3}.$$

So, with the values of $A(x), B(x), A_k\ and\ B_k$ in $P(x)$, we obtain

$$P(x) = -\frac{172}{45}x^6 + \sum_{k=0}^{\infty} (A_k + 2B_k)\ x^{2k+8}$$

It remains to show that $A_k + 2B_k < 0, \forall k = 0,1,2,\dots\dots$, which is true by **Lemma 3.3,** because $A_k + 2B_k = M(k)$.
It means every coefficient is negative in the power series expansion of $P(x)$. By differentiating it w.r.t. x, we obtain $P'(x) < 0, \forall\, x \in (0,1)$. So, $P(x)$ is monotone decreasing which implies $P(0) > P(x), for\ 0 < x$ and it shows $P(x)$ is negative.
Thus, $P(x)$ is negative decreasing on (0,1). This proves inequality (3.4).

Proof of **Theorem 2.1:**
Denote

$$\phi(x) \coloneqq \frac{\phi_1(x)}{\phi_2(x)} = \frac{ln\left(\frac{sin^{-1}x}{x}\right) + ln\left(\frac{4-x^2}{4}\right)}{x^2},$$

where $\phi_1(x) = ln\left(\frac{sin^{-1}x}{x}\right) + ln\left(\frac{4-x^2}{4}\right)\ and\ \phi_2(x) = x^2$ with $\phi_1(0+) = 0 = \phi_2(0)$.

Differentiating w.r.t. x, we obtain

$$\frac{\phi_1'(x)}{\phi_2'(x)} = \frac{1}{2}\left(\frac{1}{x(sin^{-1} x)\sqrt{1-x^2}} - \frac{4+x^2}{x^2(4-x^2)}\right) = \frac{1}{2}\phi_3(x),$$

where

$$\phi_3(x) = \frac{1}{x(sin^{-1} x)\sqrt{1-x^2}} - \frac{4+x^2}{x^2(4-x^2)},$$

Now, we show that $\phi_3(x)$ is increasing on $(0,1)$ $i.e. \phi_3'(x) > 0, \forall x \in (0,1)$. Now, differentiating w.r.t. x, we get

$$\phi_3'(x) = \frac{1}{x^2}\phi_4(x),$$

where

$$\phi_4(x) = -\frac{1}{(sin^{-1} x)^2(1-x^2)}\left(x + \frac{(1-2x^2)(sin^{-1} x)}{\sqrt{1-x^2}}\right) - \frac{2(x^4+8x^2-16)}{x(x^2-4)^2}$$

Now, $\phi_3'(x) > 0 \iff \phi_4(x) > 0$, because $\frac{1}{x^2} > 0, \forall x \in (0,1)$.

By **Lemma 3.4**, $\phi_4(x) > 0$ and hence $\phi_3'(x) > 0, \forall x \in (0,1)$. So, $\phi_3(x)$ and its scalar multiples are increasing on $(0,1)$.

Therefore, $\phi_1'(x)/\phi_2'(x)$ is increasing on $(0,1)$. So, by **Lemma 3.1**, $\phi(x)$ is increasing on $(0,1)$. This implies that $\phi(0+) < \phi(x) < \phi(1-)$ and $(0+) = -1/12$, $\phi(1-) = ln(3\pi/8)$, which proves the theorem.

Proof of **Lemma 3.5:**

Denote the R.H.S. of (3.3) by $n(k)$. We have to prove $0 > -n(k) = N(k)$ (say). Utilising **Lemma 3.2** in $N(k)$, we get

$$\begin{aligned} &N(k) \\ &= 2a_k + 15a_{k+1} + 24a_{k+2} - 16a_{k+3} + 2b_k - 18b_{k+1} - 16b_{k+2} + 32b_{k+3} \\ &= \frac{2^{2k} \times s(k)}{(2k+1)!\,(8k^7 + 140k^6 + 1022k^5 + 4025k^4 + 9212k^3 + 12215k^2 + 8658k + 2520)} \end{aligned}$$

where

$$\begin{aligned} &s(k) \\ &= -100k^7 - 1720k^6 - 11947k^5 - 43592k^4 - 90502k^3 - 107411k^2 - 68100k - 18016 \end{aligned}$$

Observe that, s(k) is negative for all $k = 0, 1, 2, \dots$....

Therefore, $N(k) < 0$ and hence $n(k) > 0, \forall\, k$. This proves the inequality (3.5).

Proof of **Lemma 3.6:**

To prove inequality (3.6), first we rewrite it and denote new form by

$$Q(x) = x^2(x^2+4)^2 + A(x) + B(x) < 0.$$

Now we prove that $Q(x)$ is negative decreasing function, where

$$C(x) = \frac{(x-2x^3)(x^2+4)^2(sin^{-1}x)}{\sqrt{1-x^2}} \; and$$
$$D(x) = 2(x^4-8x^2-16)(1-x^2)(sin^{-1}x)^2$$

Now, by utilizing **Lemma 3.2** in $C(x) and\ D(x)$, we simply get the following

$$C(x) = -\frac{337}{15}x^6 - \frac{40}{3}x^4 + 16x^2 + \sum_{k=0}^{\infty} C_k\, x^{2k+8} \; and$$
$$D(x) = \frac{397}{45}x^6 + \frac{8}{3}x^4 - 16x^2 + \sum_{k=0}^{\infty} D_k\, x^{2k+8}$$

Where

$$C_k = -2a_k - 15a_{k+1} - 24a_{k+2} + 16a_{k+3} \; and$$
$$D_k = -b_k + 9b_{k+1} + 8b_{k+2} - 16b_{k+3}.$$

So, with the values of $C(x), D(x), C_k \; and \; D_k$ in $Q(x)$, we obtain

$$Q(x) = -\frac{172}{45}x^6 + \sum_{k=0}^{\infty} (C_k + 2D_k)\; x^{2k+8}$$

It remains to show that $C_k + 2D_k < 0, \forall k = 0, 1, 2, \dots \dots$, which is true by **Lemma 3.5,** because $C_k + 2D_k = N(k)$.

It means every coefficient is negative in the power series expansion of $Q(x)$. By differentiating it w.r.t. x, we obtain $Q'(x) < 0, \forall\, x \in (0,1)$. So, $Q(x)$ is monotone decreasing which implies $Q(0) > Q(x), for\ 0 < x$ and it shows Q(x) is negative.
Thus, Q(x) is negative decreasing on $(0,1)$. This proves inequality (3.6).

Proof of **Theorem 2.2:**
Denote

$$\psi(x) := \frac{\psi_1(x)}{\psi_2(x)} = \frac{ln\left(\frac{sin^{-1}x}{x}\right) + ln\left(\frac{4+x^2}{4}\right)}{x^2},$$

where $\psi_1(x) = ln\left(\frac{sin^{-1}x}{x}\right) + ln\left(\frac{4+x^2}{4}\right) \; and \; \psi_2(x) = x^2$ with $\psi_1(0+) = 0 = \psi_2(0)$.

Differentiating w.r.t. x, we obtain

$$\frac{\psi_1'(x)}{\psi_2'(x)} = \frac{1}{2}\left(\frac{1}{x(sin^{-1}x)\sqrt{1-x^2}} + \frac{x^2-4}{x^2(4+x^2)}\right) = \frac{1}{2}\psi_3(x),$$

where

$$\psi_3(x) = \frac{1}{x(sin^{-1}x)\sqrt{1-x^2}} + \frac{x^2-4}{x^2(4+x^2)},$$

Now, we show that $\psi_3(x)$ is increasing on (0,1) $i.e. \psi_3'(x) > 0, \forall x \in (0,1)$.
Differentiating w.r.t. x, we obtain

$$\psi_3'(x) = \frac{1}{x^2}\psi_4(x),$$

where

$$\psi_4(x) = -\frac{1}{(sin^{-1}x)^2(1-x^2)}\left(x + \frac{(1-2x^2)(sin^{-1}x)}{\sqrt{1-x^2}}\right) + \frac{2(x^4-8x^2-16)}{x(x^2+4)^2}$$

Now, $\psi_3'(x) > 0 \iff \psi_4(x) > 0$, because $\frac{1}{x^2} > 0, \forall x \in (0,1)$.
By **Lemma 3.6**, $\psi_4(x) > 0$ and hence $\psi_3'(x) > 0, \forall x \in (0,1)$. So, $\psi_3(x)$ and its scalar multiples are increasing on (0,1).
Therefore, $\psi_1'(x)/\psi_2'(x)$ is increasing on (0,1). So, by **Lemma 3.1**, $\psi(x)$ is increasing on (0,1). This implies that $\psi(0+) < \psi(x) < \psi(1-)$ and $(0+) = 5/12, \psi(1-) = ln\,(5\pi/8)$, which proves the theorem.

Conclusion:

As we have noticed that the lower bound(LB) in (1.4) is sharper than that of (1.5) and (1.3). the LB for function $f(x) = sin^{-1}x/x$ in (2.2) is the refinement of LB in (1.4). Also, the LB of (2.1) is tighter than the LB in (2.2) for $f(x)$. The upper bound(UB) of it in (1.4) is very tight than that of the UBs in (1.3), (1.5), (2.1) and (2.2). Despite this fact, the UB in (2.1) is the refinement of UBs of f(x) in (1.3), (1.5) and (2.2). Moreover, the UB of it in (2.2) is tighter than that in (1.3) and (1.5). We conclude this article by writing following inequality,

$$\begin{aligned} 1 + \left(\frac{\pi-2}{2}\right)x^2 &> e^{\ln\left(\frac{\pi}{2}\right)x^2} > \left(1+\frac{x^2}{4}\right)^{-1} e^{\ln\left(\frac{5\pi}{8}\right)x^2} > \left(1-\frac{x^2}{4}\right)^{-1} e^{\ln\left(\frac{3\pi}{8}\right)x^2} \\ &> \frac{\pi}{\pi-(\pi-2)x^2} > \frac{\sin^{-1}x}{x} > \left(1-\frac{x^2}{4}\right)^{-1} e^{\frac{-x^2}{12}} > \left(1+\frac{x^2}{4}\right)^{-1} e^{\frac{-5x^2}{12}} \\ &> \frac{6}{6-x^2} > e^{\frac{x^2}{6}} > 1 + \frac{1}{6}x^2 \end{aligned}$$

Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies involving human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all participants in this study.

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