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A NOTE ON PSEUDO SYMMETRIC IDEALS OF PARTIALLY ORDERED TERNARY SEMIGROUPS

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Abstract
In this article, we study some interesting properties of pseudo symmetric ideals and prime pseudo symmetric ideals in partially ordered ternary semigroup.

Keywords: Partially ordered ternary semigroup, pseudo symmetric ideal, prime pseudo symmetric ideals.

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Introduction
Lehmer D. H. [3] studied the triplexes algebraic systems for commutative ternary groups. The ideal theory of n-ary semigroups and ternary semigroup was introduced by Sioson F. M. [5] in 1965. Dixit V. N. and Dewan S. [7] studied the properties of quasi-ideals and bi-ideals in ternary semigroups. Iampan A. [1, 2] has developed the theory of ordered ternary semigroups along the line of the theory of ordered semigroups and ternary semigroups. Daddi V. R. and Pawar Y. S. [8] defined the notion of an ordered quasi-ideal and an ordered bi-ideal in an ordered ternary semigroup. In 2014, Siva Rami Reddy V. et al. [9, 10] have established the ideal theory of a partially ordered ternary semigroup. They also defined and studied the notions of complete prime ideals, prime ideals, complete semiprime ideals and semiprime ideals of partially ordered ternary semigroups. Jyothi V. et al. in [6] introduced the notion of semipseudo symmetric ideals and pseudo symmetric ideals of partially ordered ternary semigroups. The notions of prime, semiprime pseudo symmetric ideals and irreducible pseudo symmetric ideals of partially ordered ternary semigroups is defined by Shinde D. N. and Gophane M. T in [4].

Preliminaries
A non-empty set T with a ternary operation $[\ ]: T \times T \times T \to T$ is called a ternary semigroup [3] if $[\ ]$ satisfies the associative law, $[p\,q\,r\,s\,t] = [[p\,q\,r]\,s\,t] = [p\,[q\,r\,s]\,t] = [p\,q\,[r\,s\,t]]$, for all $p, q, r, s, t \in T$.
For non-empty subsets X, Y and Z of a ternary semigroup T, $[XYZ] = \{[xyz]: x \in X, y \in Y \text{ and } z \in Z\}$. For easiness, we write, $[XYZ]$ as XYZ, $[xyz] = xyz$ and $[XXX] = X^3$.
A ternary semigroup T is said to be a partially ordered ternary semigroup [1] if there exist a partially ordered relation $\leq$ on T such that, $a \leq b \Rightarrow xya \leq xyb, xay \leq xby, axy \leq$

bxy for all $a, b, x, y \in T$. In this article, we write T for a partially ordered ternary semigroup, unless otherwise specified.
A partially ordered ternary semigroup T is said to be commutative [2] if $xyz = zxy = yzx = yxz = zyx = xzy$, for all $x, y, z \in T$.
Let X be a non-empty subset of T. We denote, $\{t \in T : t \leq x, \text{for some } x \in X\}$ by $(X]$.
A non-empty subset I of T is said to be a left (respectively, right, lateral) ideal [8] of T if $TTI \subseteq I$ (respectively, $ITT \subseteq I, TIT \subseteq I$) and $(I] = I$. A non-empty subset I of T is said to be ideal [8] of T if it is a left ideal, a right ideal and a lateral ideal of T.
An ideal I of T is said to be a pseudo symmetric ideal [6] if $x, y, z \in T, xyz \in I$ implies $xsytz \in I\ \forall\ s, t \in T$. A pseudo symmetric ideal I of T is said to be proper pseudo symmetric ideal of T if it different from T.
A proper pseudo symmetric ideal I of T is said to be a prime pseudo symmetric ideal [4] of T if $I_1 I_2 I_3 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$ where I_1, I_2, I_3 are pseudo symmetric ideals of T.

Result: [8] Let A, B and C be non-empty subset of T then the following statements hold,
(1) $A \subseteq (A]$.
(2) $((A]] = (A]$.
(3) $(A](B](C] \subseteq (ABC]$.
(4) If $A \subseteq B$ then $(A] \subseteq (B]$.

Main Results:

Pseudo Symmetric Ideals

Theorem 1: If I_1 and I_2 are two pseudo symmetric ideal of T then $I_1 \cap I_2$ is also pseudo symmetric ideal of T, provided $I_1 \cap I_2 \neq \emptyset$.

Proof: Since I_1 and I_2 are ideal of T. So, $I_1 \cap I_2$ is ideal of T. Let $xyz \in I_1 \cap I_2\ \forall\ x, y, z \in T \Rightarrow xyz \in I_1\ \&\ xyz \in I_2$. If $xyz \in I_1$ and I_1 is pseudo symmetric ideal of T then $xsytz \in I_1\ \forall\ s, t \in T$. If $xyz \in I_2$ and I_2 is pseudo symmetric ideal of T then $xsytz \in I_2\ \forall\ s, t \in T$. So, $xsytz \in I_1 \cap I_2$. Hence $I_1 \cap I_2$ is a pseudo symmetric ideal of T.

Theorem 2: Arbitrary intersection of pseudo symmetric ideal of T is a pseudo symmetric ideal of T, provided it is non-empty.

Proof: Let $\{I_i\}_{i \in \Delta}$ be a family of pseudo symmetric ideals of T. Let $I = \bigcap_{i \in \Delta} I_i \neq \emptyset$, be the intersection of this family of T. Here I is an ideal. Let $xyz \in I\ \forall\ x, y, z \in T \Rightarrow xyz \in I_i, i \in \Delta$ and I_i is pseudo symmetric ideal of T then $xsytz \in I_i, i \in \Delta\ \forall\ s, t \in T \Rightarrow xsytz \in \bigcap_{i \in \Delta} I_i,\ \forall\ s, t \in T \Rightarrow xsytz \in I\ \forall\ s, t \in T$. Hence I is a pseudo symmetric ideal of T.

Theorem 3: If I_1 and I_2 are two pseudo symmetric ideal of partially ordered ternary semigroup T then $I_1 \cup I_2$ is also pseudo symmetric ideal of T.

Proof: Since, I_1 and I_2 are ideal of T. So, $I_1 \cup I_2$ is ideal of T. Let $xyz \in I_1 \cup I_2 \;\forall\; x, y, z \in T \Rightarrow$ either $xyz \in I_1$ or $xyz \in I_2$. If $xyz \in I_1$ and I_1 is pseudo symmetric ideal of T then $xsytz \in I_1 \;\forall\; s, t \in T \Rightarrow xsytz \in I_1 \cup I_2$. If $xyz \in I_2$ and I_2 is pseudo symmetric ideal of T then $\in I_2 \;\forall\; s, t \in T \Rightarrow \quad xsytz \in I_1 \cup I_2$. Hence $I_1 \cup I_2$ is a pseudo symmetric ideal of T.

Theorem 4: Arbitrary union of pseudo symmetric ideal of T is a pseudo symmetric ideal of T.

Proof: Let $\{I_i\}_{i\in\Delta}$ be a family of pseudo symmetric ideals of T. Let $I = \bigcup_{i\in\Delta} I_i$ be the union of this family of pseudo symmetric ideal of T. Hence I is an ideal of T. Now, let $xyz \in I \;\forall\; x, y, z \in T \Rightarrow xyz \in I_i$ for some $i \in \Delta$ and I_i is pseudo symmetric ideal of T then $xsytz \in I_i$ for some $i \in \Delta \;\; \forall\; s, t \in T \Rightarrow xsytz \in \bigcup_{i\in\Delta} I_i \;\; \forall\; s, t \in T \; xsytz \in I \;\; \forall\; s, t \in T$. Hence I is a pseudo symmetric ideal of T.

Theorem 5: The collection of all pseudo symmetric ideals of partially ordered ternary semigroup T forms a poset with respect to the partial ordering relation $\subseteq$.

Proof: Let $\mathcal{I} = \{I_i \,/\, i \in \Delta,\; \Delta \text{ is any indexing set}\}$ be a family of all pseudo symmetric ideals of partially ordered ternary semigroup T.
(1) Reflexive- For any $I_i \in \mathcal{I}, i \in \Delta,\; I_i \subseteq I_i \Longrightarrow \subseteq$ is reflexive.
(2) Antisymmetric- For any $I_i, I_j \in \mathcal{I}$ and $i, j \in \Delta$. If $I_i \subseteq I_j, I_j \subseteq I_i$ then $I_i = I_j \Longrightarrow \subseteq$ is antisymmetric.
(3) Transitive- For any $I_i, I_j, I_k \in \mathcal{I}$ and $i, j, k \in \Delta$. If $I_i \subseteq I_j, I_j \subseteq I_k$ then $I_i \subseteq I_k \Longrightarrow \subseteq$ is transitive. Therefore $\mathcal{I}$ forms a poset.

Theorem 6: Let I be an ideal of commutative partially ordered ternary semigroup T, then I is a pseudo symmetric ideal.

Proof: Let T be the commutative partially ordered ternary semigroup and I be any ideal of T. Let $x, y, z \in T,\; xyz \in I$ and $s, t \in T$ then $xsytz = \; xystz = xyszt = xyzst = (xyz)st \in ITT \subseteq I \Longrightarrow xsytz \in I$. Hence I is a pseudo symmetric ideal of T.

Corollary 7: If I is an ideal of commutative partially ordered ternary semigroup T then $(\,I\,]$ is a pseudo symmetric ideal of T.

Note: In a commutative partially ordered ternary semigroup, a left, a right and a lateral ideal coincide.

Theorem 8: The intersection of a left, a right and a lateral ideal of commutative partially ordered ternary semigroup T is a pseudo symmetric ideal of T.

Proof: According to above note, a left, a right and a lateral ideal are coinciding and every ideal of commutative partially ordered ternary semigroup is pseudo symmetric ideal.

Theorem 8: If I is an ideal of T and P is a pseudo symmetric ideal of T, then $I \cap P$ is a pseudo symmetric ideal of I, providing I as a partially ordered ternary semigroup.

Proof: Since, I is an ideal of T and P is a pseudo symmetric ideal of T then $I \cap P$ is an ideal of I. Let $x, y, z \in I, xyz \in I \cap P \Longrightarrow xyz \in I$ and $xyz \in P$. If $xyz \in I$ and I is an ideal of T then, for all $s, t \in I$, consider $xsytz = (xsy)tz \in III \subseteq I$ (since I is an ideal of T) $\Longrightarrow xsytz \in I$. If $xyz \in P$ and P is a pseudo symmetric ideal of T then, for all $s, t \in I \subseteq T \Longrightarrow xsytz \in P\ \forall\ s, t \in T$. Therefore $xsytz \in I \cap P$. Hence $I \cap P$ is a pseudo symmetric ideal of I.

Theorem: Every pseudo symmetric ideal of T is a bi- ideal of T.

Proof: Let I be a pseudo symmetric ideal of T. Consider $ITITI = I(TIT)I \subseteq III \subseteq TTI \subseteq I$ (since I is a pseudo symmetric ideal of T) and $(I] \subseteq I$. Hence, I is a bi- ideal of T.

Theorem: Every pseudo symmetric ideal of T is a quasi- ideal of T.

Proof: Let I be a pseudo symmetric ideal of T. Consider $(TTI] \cap (TIT \cup TTITT] \cap (ITT] \subseteq (I] \cap (I] \cap (I] = (I] \subseteq I$. Therefore $(TTI] \cap (TIT \cup TTITT] \cap (ITT] \subseteq I$ and $(I] \subseteq I$ (since I is a pseudo symmetric ideal of T). Hence I is quasi- ideal of T.

Prime Pseudo Symmetric Ideals

Theorem: A proper pseudo symmetric ideal I of T is prime if and only if $I_1, I_2, I_3, \ldots \ldots, I_n$ are pseudo symmetric ideals of T, where n is an odd natural number, $I_1\, I_2\, I_3 \ldots \ldots I_n \subseteq I$ implies $I_i \subseteq I$ for some $i = 1, 2, 3, \ldots \ldots, n$.

Proof: Suppose I is a prime pseudo symmetric ideal of T. Let $I_1, I_2, I_3, \ldots \ldots, I_n$ be pseudo symmetric ideals of T such that $I_1 I_2 I_3 \ldots \ldots I_n \subseteq I$, where n is an odd natural number.
If $n = 1$ then $I_1 \subseteq I$.
If $n = 3$ then $I_1 I_2 I_3 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$ (since I is a prime pseudo symmetric ideal of T). Hence $I_i \subseteq I$ for some $i = 1, 2, 3$.
If $n = 5$ then $I_1 I_2 I_3 I_4 I_5 \subseteq I \Rightarrow I_1 I_2 I_3 \subseteq I$ or $I_4 \subseteq I$ or $I_5 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$ or $I_4 \subseteq I$ or $I_5 \subseteq I$ (since I is a prime pseudo symmetric ideal of T). Hence $I_i \subseteq I$ for some $i = 1, 2, 3, 4, 5$.
Therefore, by induction on n, $I_1 I_2 I_3 \ldots \ldots I_n \subseteq I \Rightarrow I_i \subseteq I$ for some $i = 1, 2, 3, \ldots \ldots, n$.
Conversely, let $I_1, I_2, I_3, \ldots \ldots, I_n$ be pseudo symmetric ideals of T, where n is an odd natural number such that $I_1 I_2 I_3 \ldots \ldots I_n \subseteq I \Rightarrow I_i \subseteq I$ for some $i = 1, 2, 3, \ldots \ldots, n$. By the definition of prime pseudo symmetric ideal, I is a prime pseudo symmetric ideal of T.

Definition: [4] A proper pseudo symmetric ideal I of T is said to be a maximal pseudo symmetric ideal of T if I is not properly contained in any proper pseudo symmetric ideal of T.

Theorem: If T is a partially ordered ternary semigroup such that $T^3 = T$ then every maximal pseudo symmetric ideal of T is a prime pseudo symmetric ideal of T.

Proof: Let I be a maximal pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be pseudo symmetric ideals of T such that $I_1\, I_2\, I_3 \subseteq I$. Suppose that $I_1 \nsubseteq I$, $I_2 \nsubseteq I$, $I_3 \nsubseteq I$. If

$I_1 \nsubseteq I \Rightarrow I_1 \cup I$ is a pseudo symmetric ideal of T and $I \subset I_1 \cup I \subseteq T$. Since I is a maximal pseudo symmetric ideal of T, $I_1 \cup I = T$. Similarly, we can prove that $I_2 \cup I = T$ and $I_3 \cup I = T$. Now, $T = T^3 = TTT = (I_1 \cup I)(I_2 \cup I)(I_3 \cup I) \subseteq I \Rightarrow T \subseteq I$. Thus $I = T$. Which is contradiction. Therefore either $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. Hence I is a prime pseudo symmetric ideal of T.

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