EFFECT OF PARETO OPTIMALITY IN MULTI OBJECTIVE OPTIMIZATION UNDER FUZZY ENVIRONMENT USING RANK AND DIVERSITY

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ABSTRACT

Study of multi objective optimization takes a major contribution in the field of research activities widely by the researchers due to the conflicting nature of ob- jectives.various works have been progressed to find details about pareto optimal solution.In continuation to this, researchers developed so many algorithms for more details about this work.Keeping inspired by Genetic algorithm, authors intend to show the consequences of pareto optimality of multi objective problems. Due to the volatile nature of decision maker, the concept of Fuzzy optimization has been introduced in correlating with the theory of optimization using several objectives. Here, authors discuss the result of pareto optimality under the influence of fuzzy parameters in terms of Rank,Diversity and Pareto efficiency. Different case studies have been analysed for more details about study of pareto optimality and it's effects in involvement of fuzzy parameters.

Keywords: Multi-objective Non Linear programming problem; Pareto Optimality; Rank and Diversity.

1. INTRODUCTION.

As per the literature review, we find the involvement of optimization with several objectives in various problems related to engineering, indus- try and other fields like agriculture.It has been widely spread due to the interesting property of conflicting objectives in incomparable units. Im- provement of one objective depends on the worst performance of other objectives which is basically the concept of multi objective optimization problems.For an example ,while purchasing a car our main concerns are cost and comfortability.We want the cost is to be minimum with maxi- mum comfortability.

In comparing with single objective optimization,we find a single opti- mal solution in detecting between pair of solutions.In case of multi objec- tive optimization, it is a difficult task to find the best among all possible solutions. The concept of pareto dominance relation is introduced to compare in between the solutions. Several research works have been done and still the research is going on to find the best method.As per initiation this method somehow helps in detecting non-dominated solu- tions by finding possible trade-off between objectives. In multi objective optimization mainly we have two concerns.

- To find multiple solutions for converging towards optimality.
- For choosing best solution by correlating judgement of decisionmaker.

In continuation with this by detecting pareto optimal solutions and to overcome difficulties,weighted sum method has been introduced to pro- duce set of multiple solutions by choosing the weights significantly.By Marler and Arora in 2004, there are variety approaches to determine weights consistently.Still this process was a benchmark for obtaining best optimal solution in choosing significant value of weights.

Despite the limitation of this method researchers started studying to find the best solution and introduced Genetic algorithm approach by Hol-land[32] and his colleagues in the year 1960. As per terminology Genetic algorithm deals with evolutionary approach

where solution vector $x \in X$is called an individual or a chromosome. Using the process of encoding of
genes over a random population we can able to say the best optimal so- lution approximately in dealing with non dominated solutions in pareto front by the theory of Rank and diversity. Using crowding or cluster- ing distance approach, we can able to detect the spreading of solution points in a certain neighbourhood, which causes more clarity towards the best optimal solutions. Maintaining diversity in population is an im- portant aspect in multi-objective GA for obtaining best among optimal solutions. In case of several objective functions, it is a very difficult task for the decision maker to maintain the aspiration level. If $\mu_j(Z_j(x^*)) = 1$ for some j, then we say x^* would be the optimal solution or the goal is achieved. Under the influence of Fuzzy parameter, it is observed that in some cases fuzzy efficiency imply pareto-optimality. Depending upon the degree of satisfaction level as one, we must observe a coherent re- lationship between fuzzy efficiency and pareto efficiency. Dealing with fuzzy parameter in optimization, fuzzy multi-objective non linear pro- gramming (FMONLP) problem was introduced with a view of restoring decisions for best optimal solutions. In order to change to Crisp problem the process of defuzzification[30] was introduced.

The paper is comprised of following sections as follows.Section-1 de- scribes introduction. In section-2 and 3, we mention notations and pre- liminaries.In section-4, we analyse the pareto optimality ,rank,diversity of the optimal solutions and check fuzzy efficiency of multi objective non linear programming proble(MONLPP).In section-5, we perform a comparative study based on observation.In section-6, we describe the conclusion.

2. NOTATIONS AND PRELIMINARIES

2.1. NOTATIONS. $z^1 \prec_{pareto} z^2$: Vector z^1 Pareto-dominates vector $z^2$$f(x) \prec_{pareto} f(x^*)$: Solution $x^* \in X$ is Pareto Optimal

$f(x) < f(x^*)$: Solution $x^* \in X$ is weakly Pareto optimal

x_1^* :Decision variable x_2^* :Decision variable w_1 :Weight function

w_2 :Weight function

A,B,C,D,E,F,G,H,I,J : Solution to the multi objective problem

L_1 : Bound with lower value U_1 : Bound with upper valueL_2 :Bound with lower value U_2 : Bound with upper value

Z_1^* :Value of 1st objective function for different x_1 ,x_2

Z_2^* : Value of 2nd objective function for different x_1 ,x_2

$\mu_{Z_1^*}(x)$:Significant value corresponds to Z_1^*

$\mu_{Z_2^*}(x)$ **:Significant value corresponds to** Z_2^*
$\tilde{Z}_1$ **:Value of 1st objective function corresponds to fuzzy parameter**
$\tilde{Z}_2$ **:Value of 2nd objective function corresponds to fuzzy parameter**
x_1 **: Decision variable**
x_2 **: Decision variable**
F(x) : Linear combination of the objective function with proper weightfunction

3. PRELIMINARIES

Pareto Dominance relation. :[31]

We say that the vector z^1 **dominates vector** z^2**,denoted by** $z^1 \prec_{pareto} z^2$**,iff** $\forall i \in 1, 2, ..., k : z^1_i \leq z^2_i$ **and** $\exists i \in 1, ..., k : z^1_i \nless z^2_i$**.**

Pareto Optimality:[31]

A solution $x^* \in X$ **is Pareto Optimal if there does not exist anothersolution x** $\in X$ **such that** $f(x) \prec_{pareto} f(x^*)$**.**

Weak Pareto Optimality:[31]

A solution $x^* \in X$ **is weakly Pareto optimal if there does not existanother solution**$x \in X$ **such that** $f(x) < f(x^*)$ **for all i=1,...,k.**

Pareto Optimal set:[31]

The Pareto optimal set,P^***,is defined as:**

$P^* = (x \in X | \$ y \in X : f(y) \leq f(x))$**.**

Pareto front:

A curve containing non dominated solutions of same rank.Fuzzy-efficient:[2]

A decision plan $x^o \in X$ **is said to be a fuzzy-efficient solution to the FMONLP if and only if \$ another** $y \in X$ **such that** $\mu_i(Z_i(y)) \geq \mu_i(Z_i(x^o))$**forall i and** $\mu_i(Z_i(y)) > \mu_i(Z_i(x^o))$ **for at least one j.**

Defuzzification of PIFN:[30]

Let $\tilde{U}^{PI} = (u_1, u_2, u_3, u_4, u_5; u^1_1, u^1_2, u^1_3, u^1_4, u^1_5)$ **be a PIFN. The crisp real number for the belongingness function** $\mu_{\tilde{U}^{PI}}$ **is denoted by** $D(\mu_{\tilde{U}^{PI}})$ **and is defined by** $D(\mu_{\tilde{U}^{PI}}) = \frac{1}{9}(u_1 + 3u2 + u_3 + 3u_4 + u_5)$**. Similarly, the crisp real number for the non-belongingness function** $v_{\tilde{U}^{PI}}$ **is denoted by** $D(v_{\tilde{U}^{PI}})$ **and is defined by** $D(v_{\tilde{U}^{PI}}) = \frac{1}{9}(u^1_1 + 3u^1_2 + u^1_3 + 3u^1_4 + u^1_5)$**. Now, the crisp real value of** $\tilde{U}^{PI}$ **can obtained taking the average of the crisp value of the belongingness function and non-belongingness function. For that we defined a ranking function of** $\tilde{U}^{PI}$ **denoted by** $\Gamma(\tilde{U}^{PI})$ **and defined**

by $\Gamma(\tilde{U}^{P\prime}) = \frac{1}{2}(D(\mu_{\tilde{U}^{PI}}) + D(\nu_{\tilde{U}^{PI}}))$
$= \frac{1}{18}((u_1 + 3u_2 + u_3 + 3u_4 + u_5) + (u_1^1 + 3u_2^1 + u_3^1 + 3u_4^1 + u_5^1))$

Pareto Efficiency:

Fuzzy efficient solution is pareto effcient.The value of the membership function is one with highest accuracy.But sometimes if it is lesser than one ,then fuzzy efficiency may converge to pareto efficiency.

4. DETAILED ANALYSIS

ANALYSIS-1. Here, we dealwith multiobjective nonlinear program-ming problem and the detail process is comprising of following steps.

Step-1:Reduce to single objective problem by using Weighted summethod.

Step-2:Using the cocept of convexity/concavity of solutions, we obtainthe value of x_1 and x_2 in terms of weights.

Step-3:Using the principle $w_1 + w_2 = 1$,we posses different values of x_1 and x_2 in relevant to the domain [0,1] of the weights.

Step-4:We find the pareto frontier corresponds to the solution pointwith the analysis of rank and diversity.

EXAMPLE-1

$MinZ_1 = 2x_1 + x_2.x_1. MinZ_2 = 2x_2^2.$

$s.t\ 3x_1 + x_2 = 3\ 4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

$$Solution : F(x) = w_1(2x_1 + x_2.x_1) + w_2(2x_2^2)$$

$$= 2w_1x_1 + w_1x_2.x_1 + 2w_2x_2^2$$

$\frac{\partial F}{\partial x_1} = 2w_1 + w_1x_2$, $\frac{\partial F}{\partial x_2} = w_1x_1 + 4w_2x_2$

Let $\frac{\partial F}{\partial x_1}$ **=0**

i,e $2w_1 + w_1x_2$**=0................(1)**

$\frac{\partial F}{\partial x_2}$ **=0**

i,e $w_1x_1 + 4w_2x_2$ **=0 .... (2)**

TABLE 1. values of objective functions

$\alpha(w_1)$	x_1^*	x_2^*	F_1	F_2
0.1	**72**	**-1.6**	**28.8**	**5.12**
0.2	**32**	**-1.2**	**25.6**	**2.88**
0.3	**18.67**	**-0.8**	**22.4**	**1.28**
0.4	**12**	**-0.4**	**19.2**	**0.32**
0.5	**8**	**0**	**16**	**0**
0.6	**5.33**	**0.4**	**12.8**	**0.32**
0.7	**3.43**	**0.8**	**9.6**	**1.28**
0.8	**2**	**1.2**	**6.4**	**2.88**
0.9	**0.89**	**1.6**	**3.2**	**5.12**
1.0	**0**	**2**	**0**	**8**

from equation-1 and equation-2

$w_1(2 + x_2) = 0$ **,** $w_1x_1 + 4w_2x_2 = 0$

$x_2^* = -2$ **,** $x_1^* = \frac{8w_2}{w_1}$

Using different value of x_1^* and x_2^* ,we get corresponding value of objective functions as mentioned in the above table.

Observation 4.1. ***RANK***

Using Dominance Rank method if we draw a rectangle taking one of the nodeas point A in the left most side towards origin ,we find there is no solution in the rectangle that means there is no solution dominates A as a result we obtain rank corresponds to the solution A is one (i.e 0+1). In this manner we get the rank of the solution from A to F is one.

At solution G, if we draw the rectangle in the left most side we find two solutionsE and F lie in that rectangle as a result we obtained rank of the solution G is two.

similarly H has rank four,I has rank 6 ,J has rank 8

Hence the solution space containing points A,B,C,D,E,F yields Pareto Frontier in the figure mentioned below.

DIVERSITY

In the figure mentioned below for detection of diversity, we use method of crowding or clustering approach.As per the method, if we draw a rectangle joining the neigh- bouring points of B i.e (A and C)in compare to the rectangle joining the neighbouring points of E i.e(D and F),we observe the diversity of the solution at point B is more than the diversity of the solution at point E . In this manner we check the diversity of solutions at different points.

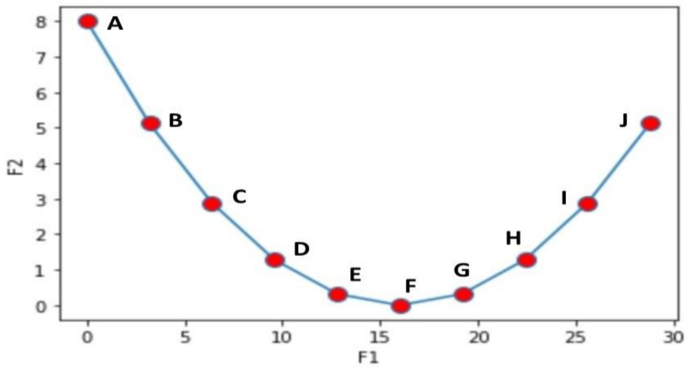

FIGURE 1. Detection of Paretofrontier .

Hence, we conclude that diversity of the solution in Pareto Frontier is more signif-icant than the diversity of the solution at the point which are not in Pareto Frontier.

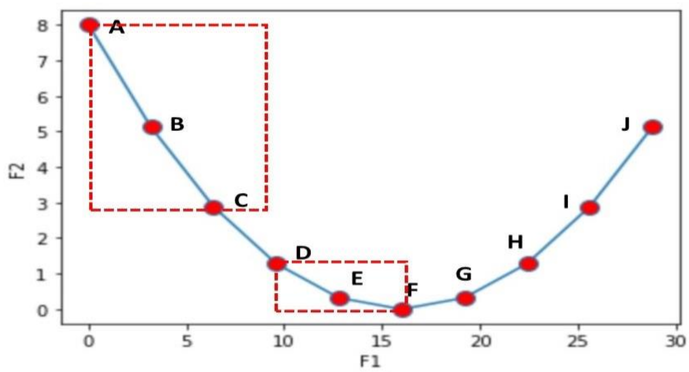

FIGURE 2. Detection of diversity .

ANALYSIS-2. Here, we dealwith multiobjective nonlinear program- ming problem under fuzzy environment and the detail process is com- prising of following steps.

EXAMPLE-2

$Min Z_1 = 2x_1 + x_2 . x_1 Min Z_2 = 2x^2$

$s.t\ 3x_1 + x_2 = 34x_1 + 3x_2 \geq 6$ 2

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

Solution:

Consider each objective function with respect to all constraints at atime and solving

For the 1st objective function the ideal solution is found

x_1**=0.3333 ;** x_2**=2.0000 ;** Z_1**=1.3333**

For the 2nd objective function the ideal solution is found

x_1**=0.6000 ;** x_2**=1.2000 ;** Z_2**=2.8800**

A Pay-off matrix is formulated as

x_1	x_2	Z_1^*	Z_2^*
0.3333	2.0000	1.3332	8.0000
0.6000	1.2000	1.9200	2.8800

Let L_1 **and** U_1 **are lower and upper bounds of** Z_1^***,** L_2 **and** U_2 **are lower and upper bounds of** Z_2^***.**

From Pay-off matrix , we found

$L_1 = 1.3332,\ U_1 = 1.9200,\ L_2 = 2.8800,\ U_2 = 8.0000.$

The membership functions of the objectives Z_1^*, Z_2^* **are defined as:**

$$\mu_{Z_1^*}(x) = \begin{cases} 0, & \text{if } Z_1^*(x) < 1.3332; \\ \dfrac{(Z_1^*(x))^t - (1.3332)^t}{(1.92)^t - (1.3332)^t}, & \text{if } 1.3332 \leq Z_1^*(x) \leq 1.92; \\ 1, & \text{if } Z_1^*(x) > 1.92. \end{cases}$$

$$\mu_{Z_2^*}(x) = \begin{cases} 0, & \text{if } Z_2^*(x) < 2.88; \\ \dfrac{(Z_2^*(x))^t - (2.88)^t}{(8.00)^t - (2.88)^t}, & \text{if } 2.88 \leq Z_2^*(x) \leq 8.00; \\ 1, & \text{if } Z_2^*(x) > 8.00. \end{cases}$$

By Zimmermann's approach the above problem reduces to

$$Max\ \lambda$$

Subject to $\mu_{U_1}(Z_1^*(x)) \geq \lambda, \mu_{U_2}(Z_2^*(x)) \geq \lambda$

$$3x_1 + x_2 = 3$$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

After simplifying with the help of membership functions, we have

$Max\ \lambda$

Subject to $(2x_1 + x_1x_2)^t - (1.3332)^t \geq \lambda(1.92)^t - (1.3332)^t\ (2x^2)^t - (2.88)^t \geq \lambda(8.00)^t - (2.88)^t$

2

$$3x_1 + x_2 = 3$$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

case-1 For t=0.25,we have

$Max\ \lambda$

Subject to $(2x_1 + x_1x_2)^{0.25} - (1.3332)^{0.25} \geq \lambda(1.92)^{0.25} - (1.3332)^{0.25}(2x^2)^{0.25} - (2.88)^{0.25} \geq \lambda(8.00)^{0.25} - (2.88)^{0.25}$

2

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The optimal solution we obtain as:

$x_1 = 0.4543, x_2 = 1.6372, \lambda = 0.5775$

$Z_1 = 1.6524, Z_2 = 5.3608$

case-2 For t=0.5 ,we have

Max λ

Subject to $(2x_1 + x_1x_2)^{0.5} - (1.3332)^{0.5} \geq \lambda(1.92)^{0.5} - (1.3332)^{0.5}(2x_2^2)^{0.5} - (2.88)^{0.5} \geq \lambda(8.00)^{0.5} - (2.88)^{0.5}$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The respective solution is:
$x_1 = 0.4517, x_2 = 1.6450, \lambda = 0.5562$
$Z_1 = 1.6464, Z_2 = 5.41205$
case-3 For t=1,we have

Max λ

Subject to $(2x_1 + x_1x_2) - (1.3332) \geq \lambda(1.92) - (1.3332)\ (2x_2^2) - (2.88) \geq \lambda(8.00) - (2.88)$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The corresponding solution is :
$x_1 = 0.4467, x_2 = 1.6600, \lambda = 0.5139$
$Z_1 = 1.6349, Z_2 = 5.5112$
case-4 For t=1.5,we have

Max λ

Subject to $(2x_1 + x_1x_2)^{1.5} - (1.3332)^{1.5} \geq \lambda(1.92)^{1.5} - (1.3332)^{1.5}(2x_2^2)^{1.5} - (2.88)^{1.5} \geq \lambda(8.00)^{1.5} - (2.88)^{1.5}$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

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Which results the optimal solution as:

$x_1 = 0.4420, x_2 = 1.6741, \lambda = 0.4726$

$Z_1 = 1.6254, Z_2 = 5.6052$

case-5 For t=2,we have

Max λ

Subject to $(2x_1 + x_1x_2)^2 - (1.3332)^2 \geq \lambda(1.92)^2 - (1.3332)^2$ $(2x_2^2)^2 - (2.88)^2 \geq \lambda(8.00)^2 - (2.88)^2$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The solution is :

$x_1 = 0.4376, x_2 = 1.6871, \lambda = 0.4328$

$Z_1 = 1.6135, Z_2 = 5.6926$

case-6 For t=2.25,we have

Max λ

Subject to $(2x_1 + x_1x_2)^{2.25} - (1.3332)^{2.25} \geq \lambda(1.92)^{2.25} - (1.3332)^{2.25}$ $(2x_2^2)^{2.25} - (2.88)^{2.25} \geq \lambda(8.00)^{2.25} - (2.88)^{2.25}$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The solution is obtained as:

$x_1 = 0.4356, x_2 = 1.6931, \lambda = 0.4137$

$Z_1 = 1.6087, Z_2 = 5.7332$

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case-7 For t=2.5,we have

Max λ

Subject to $(2x_1 + x_1x_2)^{2.5} - (1.3332)^{2.5} \geq \lambda(1.92)^{2.5} - (1.3332)^{2.5}$

$(2x^2)^{2.5} - (2.88)^{2.5}_{2} \geq \lambda(8.00)^{2.5} - (2.88)^{2.5}$ $\quad {}^{2.25} - (2.88)^{2.25} \geq \lambda(8.00)^{2.25} - (2.88)^{2.25}$ $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The solution is obtained as:

$x_1 = 0.4337, x_2 = 1.6988, \lambda = 0.3952$

$Z_1 = 1.6043, Z_2 = 5.7718$

case-8 For t=3,we have

Max λ

Subject to $(2x_1 + x_1x_2)^3 - (1.3332)^3 \geq \lambda(1.92)^3 - (1.3332)^3$ $(2x^2)^3 - (2.88)^3 \geq \lambda(8.00)^3 - (2.88)^3$

$3x_1 + x_2 = 3$ $\quad {}_2$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The solution is obtained as:

$x_1 = 0.4302, x_2 = 1.7093, \lambda = 0.3599$

$Z_1 = 1.5957, Z_2 = 5.8434$

case-9 For t=3.25,we have

Max λ

Subject to $(2x_1 + x_1x_2)^{3.25} - (1.3332)^{3.25} \geq \lambda(1.92)^{3.25} - (1.3332)^{3.25}(2x^2)^{3.25} - (2.88)^{3.25} \geq \lambda(8.00)^{3.25}$

${}_2 \quad - (2.88)^{3.25}$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The solution is obtained as:

$x_1 = 0.4286, x_2 = 1.7141, \lambda = 0.3432$

$Z_1 = 1.5919, Z_2 = 5.8763$

case-10 For t=3.5,we have

$Max\ \lambda$

Subject to $(2x_1 + x_1x_2)^{3.5} - (1.3332)^{3.5} \geq \lambda(1.92)^{3.5} - (1.3332)^{3.5}$ $(2x_2^2)^{3.5} - (2.88)^{3.5} \geq \lambda(8.00)^{3.5} - (2.88)^{3.5}$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The solution is obtained as:

$x_1 = 0.4271, x_2 = 1.7186, \lambda = 0.3272$

$Z_1 = 1.5882, Z_2 = 5.9072$

case-11 For t=4,we have

$Max\ \lambda$

Subject to $(2x_1 + x_1x_2)^4 - (1.3332)^4 \geq \lambda(1.92)^4 - (1.3332)^4$ $(2x_2^2)^4 - (2.88)^4 \geq \lambda(8.00)^4 - (2.88)^4$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$0 \leq x_1 \leq 1$

$-2 \leq x_2 \leq 2$

The solution is :

$x_1 = 0.4244, x_2 = 1.7268, \lambda = 0.2971$

$Z_1 = 1.5817, Z_2 = 5.9637$

Observation 4.2.

RANK

The figure-3 is shown below computes the rank of all solutions. By following method cited above in observation-1 we get rank of all solutions is one since none of the solution is dominated by others. The curve containing all solutions is the pareto

TABLE 2. Value of objective function for different values of t *frontier.*

t	z_1	z_2
0.25	**1.6524**	**5.3608**
0.5	**1.6464**	**5.4121**
1	**1.6349**	**5.5112**
1.5	**1.6254**	**5.6052**
2	**1.6135**	**5.6926**
2.25	**1.6087**	**5.7332**
2.5	**1.6043**	**5.7718**
3	**1.5957**	**5.8434**
3.25	**1.5919**	**5.8763**
3.5	**1.5882**	**5.9072**
4	**1.5817**	**5.9637**

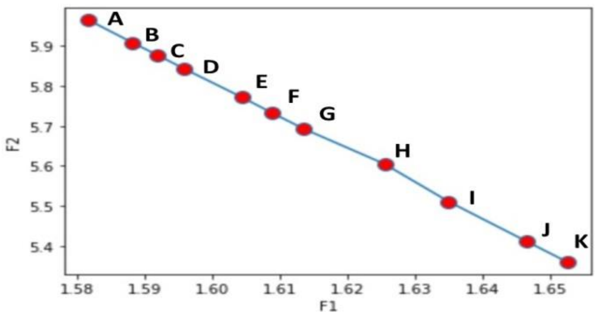

FIGURE 3. Detection of Pareto frontier.

DIVERSITY

In the figure-4 mentioned below for detection of diversity, we use method of crowd-ing or clustering approach.As per the method, if we draw a rectangle joining the neighbouring points of B i.e (A and C)in compare to the rectangle joining the neigh- bouring points of E i.e(D and F),we observe the diversity of the solution at point Eis more than the diversity of the solution at point B .

In this maner we check the diversity of solutions at different points.

Herewith we conclude that diversity at the solution points G ,H ,I J are having more diversity in compare to other solution points. This happens due to existence and degree of membership functions.

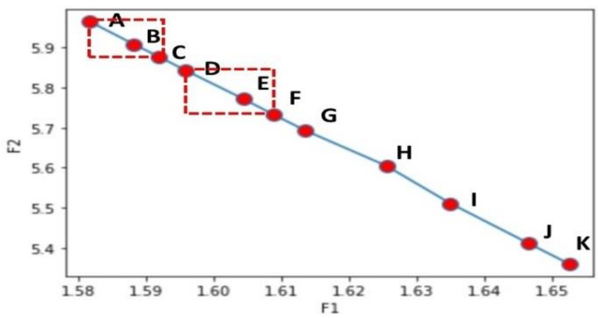

FIGURE 4. Detection of diversity .

CONVERGENCE TO PARETO EFFICIENCY

For t=0.25 , $Z_1 = 1.6524, Z_2 = 5.3608, \mu_{Z_1^*}(x) = 0.5778, \mu_{Z_2^*}(x) = 0.5774$ *For t=0.5* , $Z_1 = 1.6464, Z_2 = 5.4121, \mu_{Z_1^*}(x) = 0.5563, \mu_{Z_2^*}(x) = 0.5563$ *For t=1* , $Z_1 = 1.6349, Z_2 = 5.5112, \mu_{Z_1^*}(x) = 0.5141, \mu_{Z_2^*}(x) = 0.5139$ *For t=1.5* , $Z_1 = 1.6254, Z_2 = 5.6052, \mu_{Z_1^*}(x) = 0.4753, \mu_{Z_2^*}(x) = 0.4723$ *For t=2* , $Z_1 = 1.6135,$ $Z_2 = 5.6926, \mu_{Z_1^*}(x) = 0.4327, \mu_{Z_2^*}(x) = 0.4328$
and so on.

ANALYSIS-3. Here, we deal with multi objective non linear pro- gramming problem with fuzzy coefficients and the detail process is com-prising of following steps.

Step-1:By using defuzzification, we change the given problem to crispproblem.

Step-2: Use weighted sum method, reduce to a single objective prob-lem.

Step-3:Using the concept of convexity/concavity of solutions, we obtainthe value of x_1 and x_2 in terms of weights.

Step-4:Using the principle $w_1 + w_1 = 1$, we posses different values of x_1 and $x_2$in relevant to the domain [0,1] of the weights.

Step-5:We find the pareto frontier corresponds to the solution pointwith the analysis of rank and diversity.

EXAMPLE-3

$MinZ_1 = \tilde{2}x_1 + \tilde{1}x_2.x_1$

$MinZ_2 = \tilde{2}x_2^2$

s.t $\tilde{3}x_1 + \tilde{1}x_2 = \tilde{3}$

$\tilde{4}x_1 + \tilde{3}x_2 \geq \tilde{6}$

$\tilde{0} \leq x_1 \leq \tilde{1}$

$\tilde{-2} \leq x_2 \leq \tilde{2}$

Solution:

Where $\tilde{0} = (0.1, 0.2, 0.5, 0.4, 0.7; 0.2, 0.3, 0.6, 0.5, 0.9)$

$\tilde{1} = (0.4, 0.6, 1, 1.3, 1.7; 0.7, 0.8, 1, 1.4, 2.7)$

$\tilde{2} = (0.6, 1.9, 2, 3, 3.1; 1.3, 1.8, 2, 3.5, 3.6)$

$\tilde{3} = (1.1, 2.2, 3, 4, 4.3; 1.5, 2.4, 3, 4.1, 4.8)$

$\tilde{4} = (2.1, 3.1, 4, 4.9, 5.4; 2.2, 3.2, 4, 5.8, 6.9)$

$\tilde{6} = (4, 5, 6, 7, 8; 3, 4, 6, 8, 9)$

$-\tilde{2} = (-5, -3, -2, -1, 2; -5, -4, -2, -1, 3)$

After converted to crisp by the rule of defuzzification,we get

$MinZ_1 = 2.4x_1 + 1.1x_2.x_1$

$MinZ_2 = 2.4x_2^2$

$$s.t\ 3.1x_1 + 1.1x_2 = 3.14.2x_1 + 3.1x_2 \geq 6$$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

By Weighted sum method

$F(x) = w_1(2.4x_1 + 1.1x_2.x_1) + w_2(2.4x_2^2)$

$= 2.4w_1x_1 + 1.1w_1x_2.x_1 + 2.4w_2x_2^2$

$\frac{\partial F}{\partial x_1} = 2.4w_1 + 1.1w_1x_2$, $\frac{\partial F}{\partial x_2} = 1.1w_1x_1 + 4.8w_2x_2$

Let $\frac{\partial F}{\partial x_1} = 0$

i,e $2.4w_1 + 1.1w_1x_2$ **=0 ....(1)**

Let $\frac{\partial F}{\partial x_2} = 0$

i,e $1.1w_1x_1 + 4.8w_2x_2$ **=0..............(2)**

From equation-1

$w_1(2.4 + 1.1x_2) = 0$

$x_2^* = -2.18$

$\alpha(w_1)$	x_1^*	x_2^*	F_1	F_2
0.1	**85.59**	**-1.56**	**58.54**	**5.84**
0.2	**38.04**	**-1.12**	**44.43**	**3.01**
0.3	**22.19**	**-0.68**	**36.66**	**1.11**
0.4	**14.27**	**-0.24**	**30.48**	**0.14**
0.5	**9.51**	**0.2**	**24.92**	**0.1**
0.6	**6.34**	**0.64**	**19.68**	**0.98**
0.7	**4.08**	**1.08**	**14.64**	**2.80**
0.8	**2.38**	**1.52**	**9.69**	**5.54**
0.9	**1.06**	**1.96**	**4.83**	**9.22**
1.0	**0**	**2.4**	**0**	**13.82**

TABLE 3. Values of objective functions

From equation-2

$$1.1w_1x_1 + 4.8w_2x_2 = 0$$

$$x_1^* = 9.51\frac{w_2}{w_1}$$

Observation 4.3.

RANK

In this figure-5, using Dominance Rank Method as mentioned in analysis-1,we obtain the rank of solution points A,B,C,D,E,F,G is one.But H has rank 3,I hasrank 5,j has rank 7.

DIVERSITY

In this figure-6, using the concept of crowding or clustering approach as mentionedin analysis-1,we obtain B has more diversity than E.

ANALYSIS-4. Here, we deal with multi objective non linear pro- gramming problem with fuzzy coefficients and the Process of defuzzifica-tion.

EXAMPLE-4

$MinZ_1 = \tilde{2}x_1 + \tilde{1}x_2.x_1$

$MinZ_2 = \tilde{2}x_2^2$

s.t $\tilde{3}x_1 + \tilde{1}x_2 = \tilde{3}$

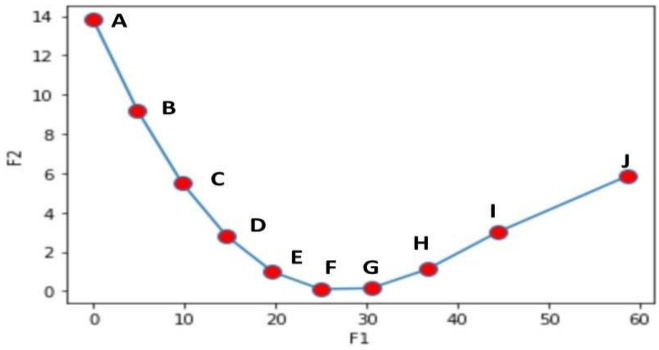

FIGURE 5. Detection of Paretofrontier.

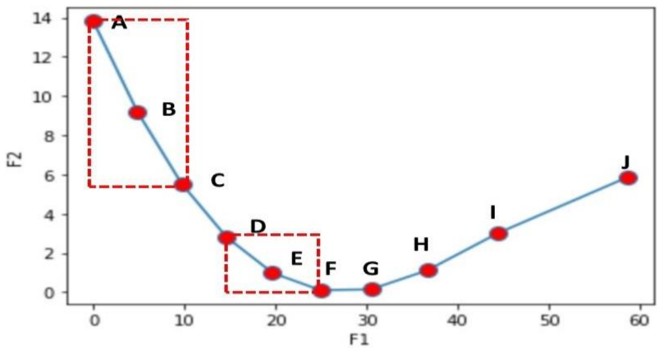

FIGURE 6. Detection of diversity .

$\tilde{4}x_1 + \tilde{3}x_2 \geq \tilde{6}$

$\tilde{0} \leq x_1 \leq \tilde{1}$

$\tilde{-2} \leq x_2 \leq \tilde{2}$

Solution:

Where $\tilde{0} = (0.1, 0.2, 0.5, 0.4, 0.7; 0.2, 0.3, 0.6, 0.5, 0.9)$

$\tilde{1} = (0.4, 0.6, 1, 1.3, 1.7; 0.7, 0.8, 1, 1.4, 2.7)$

$\tilde{2} = (0.6, 1.9, 2, 3, 3.1; 1.3, 1.8, 2, 3.5, 3.6)$

$\tilde{3} = (1.1, 2.2, 3, 4, 4.3; 1.5, 2.4, 3, 4.1, 4.8)$

$\tilde{4} = (2.1, 3.1, 4, 4.9, 5.4; 2.2, 3.2, 4, 5.8, 6.9)$

$\tilde{6} = (4, 5, 6, 7, 8; 3, 4, 6, 8, 9)$

$-\tilde{2} = (-5, -3, -2, -1, 2; -5, -4, -2, -1, 3)$

After converted to crisp by the rule of defuzzification,we get

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$MinZ_1 = 2.4x_1 + 1.1x_1x_2$

$MinZ_2 = 2.4x^2_2$

s.t $3.1x_1 + 1.1x_2 = 3.14.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

Consider each objective function with respect to all constraints at a timeand solving.

Using first objective function, the respective solution is

$x_1 = 0.4000$ x_2**=1.6909** Z_1^***=1.7040**

Using second objective function the respective solution is

x_1**=0.6032** x_2**=1.1182** Z_2^***=3.0011**

x_1	x_2	Z_1^*	Z_2^*
0.4000	1.6909	2.1896	3.0009
0.6032	1.1182	1.7040	6.8619

Let L_1 **and** U_1 **are bounds with lower value and upper value of** Z_1^***,** L_2 **and** U_2 **are bounds with lower value and upper value of** Z_2^***. From Table 1, we found**

$L_1 = 1.7040,\ U_1 = 2.1896,\ L_2 = 3.0009,\ U_2 = 6.8619.$

Corresponding membership functions for Z_1^*, Z_2^* **are defined as:**

$$\mu_{Z_1^*}(x) = \begin{cases} 0, \text{ if } Z_1^*(x) < 1.7040; \\ \dfrac{(Z_1^*(x))^t - (1.7040)^t}{(2.1896)^t - (1.7040)^t}, \text{ if } 1.7040 \leq Z_1^*(x) \leq 2.1896; \\ 1, \text{ if } Z_1^*(x) > 2.1896. \end{cases}$$

$$\mu_{Z_2^*}(x) = \begin{cases} 0, \text{ if } Z_2^*(x) < 3.0009; \\ \dfrac{(Z_2^*(x))^t - (3.0009)^t}{(6.8619)^t - (3.0009)^t}, \text{ if } 3.0009 \leq Z_2^*(x) \leq 6.8619; \\ 1, \text{ if } Z_2^*(x) > 6.8619 . \end{cases}$$

By Zimmermann's approach the above problem reduces to

$$Max\ \lambda$$

Subject to $\mu_{U_1}(Z_1^*(x)) \geq \lambda, \mu_{U_2}(Z_2^*(x)) \geq \lambda$

$3.1x_1 + 1.1x_2 = 3.1$

$4.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

After simplifying with the help of membership functions, we have

$Max\ \lambda$

Subject to $(2.4x_1 + 1.1x_1x_2)^t - (1.7040)^t \geq \lambda(2.1896)^t - (1.7040)^t\ (2.4x_2^2)^t - (3.0009)^t \geq \lambda(6.8619)^t - (3.0009)^t$

$3.1x_1 + 1.1x_2 = 3.1$

$4.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

case-1 For t=0.25,we have

$Max\ \lambda$

Subject to $(2.4x_1 + 1.1x_1x_2)^{0.25} - (1.7040)^{0.25} \geq \lambda(2.1896)^{0.25} - (1.7040)^{0.25}(2.4x_2^2)^{0.25} - (3.0009)^{0.25} \geq \lambda(6.8619)^{0.25} - (3.0009)^{0.25}$

$3.1x_1 + 1.1x_2 = 3.1$

$4.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

Which results the solution as: $x_1 = 0.4951, x_2 = 1.4229, \lambda = 0.5574\tilde{Z}_1 = 1.9632, \tilde{Z}_2 = 4.6361$

case-2 For t=0.5,we have

$$Max\ \lambda$$

$$Subject\ to\ (2.4x_1 + 1.1x_1x_2)^{0.5} - (1.7040)^{0.5} \geq \lambda(2.1896)^{0.5} - (1.7040)^{0.5}(2.4x_2^2)^{0.5} - (3.0009)^{0.5} \geq \lambda(6.8619)^{0.5} - (3.0009)^{0.5}$$

$3.1x_1 + 1.1x_2 = 3.1$

$4.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

Now the solution is:

$x_1 = 0.4934, x_2 = 1.4278, \lambda = 0.5406$

$\tilde{Z}_1 = 1.9591, \tilde{Z}_2 = 4.8926$

case-3 For t=1,we have

$Max\ \lambda$

$Subject\ to\ (2.4x_1 + 1.1x_1x_2) - (1.7040) \geq \lambda(2.1896) - (1.7040)\ (2.4x_2^2) - (3.0009) \geq \lambda(6.8619) - (3.0009)$

$3.1x_1 + 1.1x_2 = 3.1$

$4.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

It yields the solution as:

$x_1 = 0.4899, x_2 = 1.4376, \lambda = 0.5074$

$\tilde{Z}_1 = 1.9505, \tilde{Z}_2 = 4.9601$

t	z_1	z_2
0.25	**1.9632**	**4.6361**
0.5	**1.9591**	**4.8926**
1	**1.9505**	**4.9601**
1.5	**1.9420**	**5.0251**
2	**1.9343**	**5.0870**

TABLE 4. Value of objective function for different value of t

case-4 For t=1.5,we have

Max λ

Subject to $(2.4x_1 + 1.1x_1x_2)^{1.5} - (1.7040)^{1.5} \geq \lambda(2.1896)^{1.5} - (1.7040)^{1.5}(2.4x^2)^{1.5} - (3.0009)^{1.5} \geq$

$_2$ $\lambda(6.8619)^{1.5} - (3.0009)^{1.5}$

$3.1x_1 + 1.1x_2 = 3.1$

$4.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

The solution is obtained as:
$x_1 = 0.4865, x_2 = 1.4470, \lambda = 0.4748$
$\tilde{Z}_1 = 1.9420, \tilde{Z}_2 = 5.0251$ **case-5 For t=2,we have**

Max λ

Subject to $(2.4x_1 + 1.1x_1x_2)^2 - (1.7040)^2 \geq \lambda(2.1896)^2 - (1.7040)^2\,(2.4x^2)^2 - (3.0009)^2 \geq \lambda(6.8619)^2 -$

$_2$ $(3.0009)^2$

$3.1x_1 + 1.1x_2 = 3.1$

$4.2x_1 + 3.1x_2 \geq 6$

$0.4 \leq x_1 \leq 1.1$

$-2 \leq x_2 \leq 2.4$

The solution is herewith:

$x_1 = 0.4834, x_2 = 1.4559, \lambda = 0.4431$

$\tilde{Z}_1 = 1.9343, \tilde{Z}_2 = 5.0870$

Observation 4.4.*RANK**The figure-7 shown below computes the rank of all solutions. By following methodcited above in observation-1 we get rank of all solutions is one since none of the solution is dominated by others. The curve containing all solutions is the pareto frontier.***

s

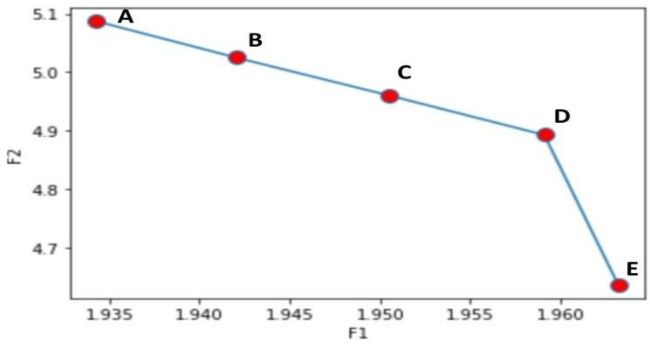

FIGURE 7. Detection of Paretofrontier.

DIVERSITY

In this figure-8 by using the method of clustering and crowding distance as men-tioned above we obtain, D has more diversity than B.

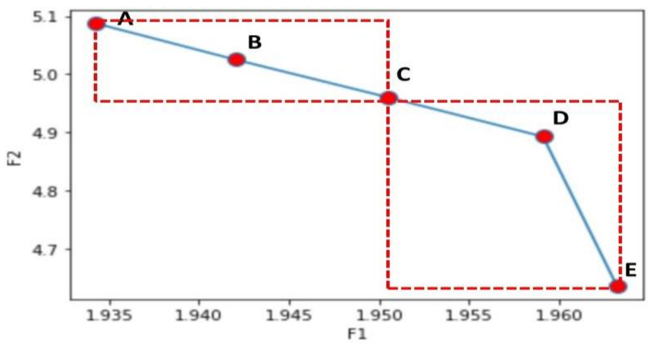

FIGURE 8. Detection of diversity.

CONVERGENCE TO PARETO EFFICIENCY

For t=0.25 , $Z_1 = 1.9632, Z_2 = 4.6361, \mu_{Z_1^*}(x) = 0.5575, \mu_{Z_2^*}(x) = 0.5001$

For t=0.5 , $Z_1 = 1.9591, Z_2 = 4.8926, \mu_{Z_1^*}(x) = 0.5409, \mu_{Z_2^*}(x) = 0.5406$ *For t=1* , $Z_1 = 1.9505, Z_2 = 4.9601, \mu_{Z_1^*}(x) = 0.5076, \mu_{Z_2^*}(x) = 0.5074$ *For t=1.5* , $Z_1 = 1.9420, Z_2 = 5.0251, \mu_{Z_1^*}(x) = 0.4745, \mu_{Z_2^*}(x) = 0.4748$ *For t=2* , $Z_1 = 1.9343, Z_2 = 5.0870, \mu_{Z_1^*}(x) = 0.4432, \mu_{Z_2^*}(x) = 0.4431$
and so on.

5. COMPARATIVE STUDY BASED ON OBSERVATION:

In this work, we have four number of analysis where analysis-1 has a significant relation with analysis-3, how ever analysis-2 has a significant relation with analysis-4.In the sense of pareto frontier by comparing fig-1and fig-5, it is found that analysis-1 has more clarity than analysis-3 but as per expectation we can over come the limitation of pareto frontier using fuzzy environment as mentioned in analysis-3,this happens due to the limitation of definition of membership function for which degree of the membership function takes major in creating clarity towards pareto frontier.Here we deal with pentagonal intuitionistic fuzzy number.

In the sense of obtaining rank of solutions in pareto frontier, we have more improvement in analysis-3 associated with fuzzy environment.

In case of diversity, we observe that the diversity of solution B is morein analysis-1 than analysis-3 also diversity of E is more in analysis-1 than analysis-3.In similar manner, we compare diversity of all solutions in analysis-1 with analysis-3.It is observed that all solutions related to analysis-1 have more diversity than analysis-3.This happens due to in-volvement of fuzzy parameter.

In the sense of having pareto efficiency in analysis-2 and analysis-4, we find $\max|\mu_{Z_1^*}(x) - \mu_{Z_2^*}(x)|$ for different values of t as 0.0004 and 0.0574. Maintaining the degree of aspiration level upto 1, we get maximum con-
vergence in analysis-4 due to involvement of fuzzy parameter.

6. CONCLUSION:

In the whole work, authors focus in the variation and effect of pareto optimality significantly by the involvement of fuzzy parameters.it is con- cluded from the layout of whole work that, the pareto frontier with the relevant properties is improved due to involvement of fuzzy parameter in terms of rank, diversity and pareto efficiency.In our future work, we willhave more studies and results regarding optimality for multi objective fractional optimization under fuzzy domain.

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