

# SEVEN POINTS COSINE RUNGE-KUTTA METHODS FOR SOLVING FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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## ABSTRACT

We present implicit Runge-Kutta method using cosine functions for solving first order ordinary differential equations. Cosine functions are used to obtain special points which are used to construct the high order implicit Runge-Kutta methods. Collocation approach at these special points are used to generate continuous schemes for the generation of discrete schemes. The discrete schemes are reformulated to Runge-Kutta function-evaluations for solution of first order ordinary differential equations. Numerical experiments are used to show that the method are more efficient, simpler and convergent to exact solutions faster and better than exiting methods.

**Keywords:** cosine function, special points, collocation method, Runge-Kutta method,

## INTRODUCTION

Runge-kutta methods are useful for solving non-linear stiff and oscillatory differential equation. A number of numerical methods for oscillatory problems have not fully been developed. Many of these methods (e.g Runge-Kutta for non-stiff problem etc). Some of them can however take advantage of special properties of the solution that maybe known in advance. What are harder to implement are type-insensitive integrators that can adaptively switch between different modes of the solution (stiff, non-stiff, oscillatory).we describe a new class of such method. We use generalized collocation techniques based on fitting the special points of cosine function by transforming the points to implicits Runge-Kutta methods for solving ordinary differential equations problems. The coefficients of the methods are functions of the frequency and the step-size. We give the general form of their coefficients, their stability and consistency.

## METHODOLOGY

A collocation method is defined in the interval  $[x_{k-1}, x_k]$  by a continuous scheme, from the initial value problem of ordinary differential equation.

$$y' = f(x, y), y(x_0) = y_0 \quad a \leq x \leq b \quad 1.1$$

We assume our approximate solution to be in form of

$$y(x) = \sum_{j=0}^{t-1} d_j(x)y_{n+1} + h \sum_{j=0}^{m-1} v_j(x)f(\bar{x}_j, y(\bar{x}_j)) \quad 1.2$$

Where  $t$  denote the number of interpolation points  $x_{n+j}$ , ( $j = 0, 1, \dots, t-1$ ) and  $m$  denotes the distinct collocation points  $\bar{x}_j$  ( $j = 0, \dots, m-1$ ),  $f(x, y)$  is continuous and differentiable. The numerical coefficients  $d_j$  ( $j = 0, 1, \dots, k-1$ ) and  $hv_j$  ( $j = 0, 1, \dots, m-1$ ) are elements of the  $(t+m) \times (t+m)$  square matrix  $A$ . They are selected so that high accurate approximation of well-behaved problem of initial value problem are obtained efficiently.

The  $d_j(x)$  and  $v_j(x)$  in (1.2) can be represented by polynomial of the form

$$d_j(x) = \sum_{i=0}^{t+m-1} d_{j,i+1} x^i, (j = 0, 1, \dots, t-1) \quad 1.3$$

$$hv_j(x) = \sum_{i=0}^{t+m-1} hv_{j,i+1} x^i, (j = 0, 1, \dots, m-1) \quad 1.4$$

With the constant coefficients  $d_{j,i+1}$  and  $hv_{j,i+1}$  to be determined. Substituting (3.3) and (3.4) into (3.2) we have

$$y(x) = \sum_{j=0}^{t-1} \sum_{i=0}^{t+m-1} d_{j,i+1} x^i y_{n+j} + h \sum_{j=0}^{m-1} \sum_{i=0}^{t+m-1} v_{j,i+1} x^i f_{n+j}$$

$$y(x) = \sum_{j=0}^{t+m-1} \{ \sum_{i=0}^{t-1} d_{j,i+1} y_{n+j} + \sum_{i=0}^{m-1} hv_{j,i+1} f_{n+j} \} x^i \quad 1.5$$

And let

$$a_j = (\sum_{i=0}^{t-1} d_{j,i+1} y_{n+j} + \sum_{i=0}^{m-1} hv_{j,i+1} f_{n+j}) \quad a_j \in R^i \quad i = 0, 1, \dots, t+m-1 \quad 1.6$$

Such that (3.5) reduces to the power series of a single variable  $x$  in the form

$$p(x) = \sum_{j=0}^{\infty} a_j x^j \quad 1.7$$

And (3.7) is used as the basis or trial function to produce an approximate solution to IVP as

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^j \quad 1.8$$

Where  $a_j \in R^j \quad j = 0, 1, \dots, t+m-1$ , and  $U \in c^m(a, b)cp(x)$

Thus equation (3.5) can be expressed explicitly in matrix form as follows

$$y(x) = (y_n, y_{n+1}, \dots, y_{n+k-1}, f_n, f_{n+1}, \dots, f_{n+m-1}) A^T B$$

$$A = \begin{pmatrix} d_{0,1} & d_{1,1} & \dots & d_{t-1,1} & hv_{0,1} & hv_{1,1} & \dots & hv_{m-1,1} \\ d_{0,2} & d_{1,2} & \dots & d_{t-1,2} & hv_{0,2} & hv_{1,2} & \dots & hv_{m-1,2} \\ d_{0,3} & d_{1,3} & \dots & d_{t-1,3} & hv_{0,3} & hv_{1,3} & \dots & hv_{m-1,3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{0,t+m} & d_{1,t+m} & \dots & d_{t-1,t+m} & hv_{0,t+m} & hv_{1,t+m} & \dots & hv_{m-1,t+m} \end{pmatrix} \quad 1.9$$

With B defined as

$$B = \begin{pmatrix} 1 & x_n & x_n^2 & \dots & x_n^{t+m-1} \\ 0 & 1 & 2x_{n+c_1} & \dots & (t+m-1)x_{n+c_1}^{t+m-2} \\ 0 & 1 & 2x_{n+c_2} & \dots & (t+m-1)x_{n+c_2}^{t+m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+c_{m-1}} & \dots & (t+m-1)x_{n+c_{m-1}}^{t+m-2} \end{pmatrix} \quad 1.10$$

Where  $c_j (j = 1, 2, \dots, m-1)$  are collocation points chosen from the special points. Matrix B is called the multistep collocation matrix which has a very simple structure and of dimension  $(t+m) \times (t+m)$  are the constant coefficients of the polynomials given in (1.6)

### Theorem

Let I denote  $(t+m) \times (t+m)$  identity matrix. Matrices A and B defined by (1.9) and (1.10) satisfy the following conditions

1.  $AB=I \leftrightarrow B=A^{-1}$
2.  $y(x) = \sum_{j=0}^{t+m-1} a_j x^j$  [Onumayi et al 1994]

### 3.2 Derivation of seven point cosine function in $[0, \pi]$

Points of cosine functions are the special values of  $\cos\theta$  in  $[0, \pi]$  where the  $\theta$  are:  $0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$  and  $\pi$ , such that

$$\cos(0) = 1, \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{3}\right) = 1/2, \cos\left(\frac{\pi}{2}\right) = 0, \cos\left(\frac{2\pi}{3}\right) = -1/2,$$

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \cos(\pi) = -1$$

**3.5 Special points:** special points are the transformed point of a cosine functions form  $[-1, 1]$  onto interval  $[0, 1]$  by a linear transformation.

$$T(x_i) = \frac{1}{2}(1 \pm x_i) \quad [Agam (2015)].$$

Where  $x_i$  are the values of cosine functions. Given by

$$\cos(0) = 1, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \cos\left(\frac{\pi}{2}\right) = 0, \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \cos(\pi) = -1,$$

$$\text{Where } x_1 = 1, x_2 = \frac{\sqrt{2}}{2}, x_3 = \frac{1}{2}, x_4 = 0, x_5 = -\frac{1}{2}, x_6 = -\frac{\sqrt{2}}{2}, x_7 = -1$$

$$T(0) = \frac{1}{2}(1 + 1) = 1, \quad T\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{\sqrt{2}}{4}, \quad T\left(\frac{1}{2}\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4},$$

$$T(0) = \frac{1}{2}(1 + 0) = \frac{1}{2}, \quad T\left(-\frac{1}{2}\right) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}, \quad T\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{\sqrt{2}}{4}$$

$$T(-1) = \frac{1}{2}(1 - 1) = 0$$

The new special points after arranging it in ascending order are now

$$p_1 = 0, p_2 = \frac{1}{2} - \frac{\sqrt{2}}{4}, p_3 = \frac{1}{4}, p_4 = \frac{1}{2}, p_5 = \frac{3}{4}, p_6 = \frac{1}{2} + \frac{\sqrt{2}}{4}, p_7 = 1$$

We assume a power series solution of six points of the form

$$y(x) = \sum_{j=0}^7 d_j x^j, y'(x) = \sum_{j=0}^7 j d_j x^{j-1}$$

Interpolation at  $x_n$  and collocate at  $x_n = x_n + p_i$ , where  $i = 0, 1, \dots, 7$ , yield a system of simultaneous equation of the form

$$y_n = d_0 + d_1 x_n + d_2 x_n^2 + d_3 x_n^3 + d_4 x_n^4 + d_5 x_n^5 + d_6 x_n^6 + d_7 x_n^7$$

$$y'_{n+p_1} = f_{n+p_1} = 0 + d_1 + 2d_2 x_{n+p_1} + 3d_3 x_{n+p_1}^2 + 4d_4 x_{n+p_1}^3 + 5d_5 x_{n+p_1}^4 + 6d_6 x_{n+p_1}^5 + 7d_7 x_{n+p_1}^6$$

$$y'_{n+p_7} = f_{n+p_7} = 0 + d_1 + 2d_2 x_{n+p_6} + 3d_3 x_{n+p_6}^2 + 4d_4 x_{n+p_6}^3 + 5d_5 x_{n+p_6}^4 + 6d_6 x_{n+p_6}^5 + 7d_7 x_{n+p_6}^6$$

1.11

where  $d_j$  are to be determined.

Thus, (1.11) can be rewritten in matrix form as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_{n+p_1} & 3x_{n+p_1}^2 & 4x_{n+p_1}^3 & 5x_{n+p_1}^4 & 6x_{n+p_1}^5 & 7x_{n+p_1}^6 \\ 0 & 1 & 2x_{n+p_2} & 3x_{n+p_2}^2 & 4x_{n+p_2}^3 & 5x_{n+p_2}^4 & 6x_{n+p_2}^5 & 7x_{n+p_2}^6 \\ 0 & 1 & 2x_{n+p_3} & 3x_{n+p_3}^2 & 4x_{n+p_3}^3 & 5x_{n+p_3}^4 & 6x_{n+p_3}^5 & 7x_{n+p_3}^6 \\ 0 & 1 & 2x_{n+p_4} & 3x_{n+p_4}^2 & 4x_{n+p_4}^3 & 5x_{n+p_4}^4 & 6x_{n+p_4}^5 & 7x_{n+p_4}^6 \\ 0 & 1 & 2x_{n+p_5} & 3x_{n+p_5}^2 & 4x_{n+p_5}^3 & 5x_{n+p_5}^4 & 6x_{n+p_5}^5 & 7x_{n+p_5}^6 \\ 0 & 1 & 2x_{n+p_6} & 3x_{n+p_6}^2 & 4x_{n+p_6}^3 & 5x_{n+p_6}^4 & 6x_{n+p_6}^5 & 7x_{n+p_6}^6 \\ 0 & 1 & 2x_{n+p_7} & 3x_{n+p_7}^2 & 4x_{n+p_7}^3 & 5x_{n+p_7}^4 & 6x_{n+p_7}^5 & 7x_{n+p_7}^6 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{pmatrix} = \begin{pmatrix} y_n \\ f_{n+p_1} \\ f_{n+p_2} \\ f_{n+p_3} \\ f_{n+p_4} \\ f_{n+p_5} \\ f_{n+p_6} \\ f_{n+p_7} \end{pmatrix}$$

i.e

BA=Y, where

$$B = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_{n+p_1} & 3x_{n+p_1}^2 & 4x_{n+p_1}^3 & 5x_{n+p_1}^4 & 6x_{n+p_1}^5 & 7x_{n+p_1}^6 \\ 0 & 1 & 2x_{n+p_2} & 3x_{n+p_2}^2 & 4x_{n+p_2}^3 & 5x_{n+p_2}^4 & 6x_{n+p_2}^5 & 7x_{n+p_2}^6 \\ 0 & 1 & 2x_{n+p_3} & 3x_{n+p_3}^2 & 4x_{n+p_3}^3 & 5x_{n+p_3}^4 & 6x_{n+p_3}^5 & 7x_{n+p_3}^6 \\ 0 & 1 & 2x_{n+p_4} & 3x_{n+p_4}^2 & 4x_{n+p_4}^3 & 5x_{n+p_4}^4 & 6x_{n+p_4}^5 & 7x_{n+p_4}^6 \\ 0 & 1 & 2x_{n+p_5} & 3x_{n+p_5}^2 & 4x_{n+p_5}^3 & 5x_{n+p_5}^4 & 6x_{n+p_5}^5 & 7x_{n+p_5}^6 \\ 0 & 1 & 2x_{n+p_6} & 3x_{n+p_6}^2 & 4x_{n+p_6}^3 & 5x_{n+p_6}^4 & 6x_{n+p_6}^5 & 7x_{n+p_6}^6 \\ 0 & 1 & 2x_{n+p_7} & 3x_{n+p_7}^2 & 4x_{n+p_7}^3 & 5x_{n+p_7}^4 & 6x_{n+p_7}^5 & 7x_{n+p_7}^6 \end{pmatrix}$$

$$A = (d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7)^T,$$

$$Y = (y_n, f_{n+p_1}, f_{n+p_2}, f_{n+p_3}, f_{n+p_4}, f_{n+p_5}, f_{n+p_6}, f_{n+p_7})^T$$

Using maple mathematical software we obtain the continuous scheme of the form

$$y(x) = y_n + h(f_{n+p_1} + f_{n+p_2} + f_{n+p_3} + f_{n+p_4} + f_{n+p_5} + f_{n+p_6} + f_{n+p_7})$$

Now evaluating the continuous scheme at  $p_1 = (0)$ ,  $p_2 = (\frac{1}{2} - \frac{\sqrt{2}}{4})$ ,  $p_3 = \frac{1}{4}$ ,  $p_4 = \frac{1}{2}$ ,  $p_5 = \frac{3}{4}$ ,  $p_6 = (\frac{1}{2} + \frac{\sqrt{2}}{4})$ ,  $p_7 = 1$

We obtain the discrete schemes:

$$y_{n+p_1} = y_n$$

$$y_{n+p_2} = y_n + \left(\frac{53}{1120} - \frac{\sqrt{2}}{2520}\right)hf_{n+p_1} + \left(\frac{2}{21} + \frac{31\sqrt{2}}{672}\right)hf_{n+p_2} + \left(\frac{29}{1260} - \frac{22\sqrt{2}}{315}\right)hf_{n+p_3} \\ + \left(\frac{17}{105} - \frac{41\sqrt{2}}{420}\right)hf_{n+p_4} + \left(\frac{11}{140} - \frac{22\sqrt{2}}{315}\right)hf_{n+p_5} + \left(\frac{2}{21} - \frac{13\sqrt{2}}{224}\right)hf_{n+p_6} \\ + \left(-\frac{13}{10080} - \frac{\sqrt{2}}{2520}\right)hf_{n+p_7}$$

$$y_{n+p_3} = y_n + \left(\frac{1829}{40320}\right)hf_{n+p_1} + \left(\frac{37}{336} + \frac{9\sqrt{2}}{128}\right)hf_{n+p_2} + \left(-\frac{41}{2520}\right)hf_{n+p_3} + \left(\frac{31}{1680}\right)hf_{n+p_4} + \\ \left(-\frac{41}{2520}\right)hf_{n+p_5} + \left(\frac{37}{336} - \frac{9\sqrt{2}}{128}\right)hf_{n+p_6} + \left(-\frac{61}{40320}\right)hf_{n+p_7}$$

$$y_{n+p_4} = y_n + \left(\frac{16}{315}\right)hf_{n+p_1} + \left(\frac{2}{21} + \frac{\sqrt{2}}{24}\right)hf_{n+p_2} + \left(\frac{17}{105}\right)hf_{n+p_3} + \left(\frac{17}{105}\right)hf_{n+p_4} + \left(-\frac{19}{315}\right)hf_{n+p_5} + \\ \left(\frac{2}{21} - \frac{\sqrt{2}}{24}\right)hf_{n+p_6} + \left(\frac{1}{210}\right)hf_{n+p_7}$$

$$y_{n+p_5} = y_n + \left(\frac{213}{4480}\right)hf_{n+p_1} + \left(\frac{9}{112} + \frac{9\sqrt{2}}{128}\right)hf_{n+p_2} + \left(\frac{33}{280}\right)hf_{n+p_3} + \left(\frac{171}{560}\right)hf_{n+p_4} + \\ \left(\frac{33}{280}\right)hf_{n+p_5} + \left(\frac{9}{112} - \frac{9\sqrt{2}}{128}\right)hf_{n+p_6} + \left(\frac{3}{448}\right)hf_{n+p_7}$$

$$y_{n+p_6} = y_n + \left(\frac{53}{1120} + \frac{\sqrt{2}}{2520}\right)hf_{n+p_1} + \left(\frac{2}{21} + \frac{13\sqrt{2}}{224}\right)hf_{n+p_2} + \left(\frac{29}{1260} + \frac{22\sqrt{2}}{315}\right)hf_{n+p_3} \\ + \left(\frac{17}{105} + \frac{41\sqrt{2}}{420}\right)hf_{n+p_4} + \left(\frac{11}{140} + \frac{22\sqrt{2}}{315}\right)hf_{n+p_5} + \left(\frac{2}{21} - \frac{31\sqrt{2}}{672}\right)hf_{n+p_6} \\ + \left(-\frac{13}{10080} + \frac{\sqrt{2}}{2520}\right)hf_{n+p_7}$$

$$y_{n+p_7} = y_n + \left(\frac{29}{630}\right) hf_{n+p_1} + \left(\frac{4}{21}\right) hf_{n+p_2} + \left(\frac{32}{315}\right) hf_{n+p_3} + \left(\frac{34}{105}\right) hf_{n+p_4} + \left(\frac{32}{315}\right) hf_{n+p_5} + \left(\frac{4}{21}\right) hf_{n+p_6} + \left(\frac{29}{630}\right) hf_{n+p_7} \quad 1.12$$

The discrete schemes (3.22) must satisfy the (1.3), to change to Runge-Kutta formula,

Hence

$$y'_{n+p_1} = f(x_{n+p_1}, y_{n+p_1}), p_1 = 0$$

$$y'_{n+p_1} = f(x_{n+p_1}, y_n + 0) = f(x_{n+p_1}, y_n)$$

$$y'_{n+p_2} = f(x_{n+p_2}, y_{n+p_2})$$

$$y'_{n+p_2} = f\left(x_{n+p_2}, y_n + \left(\frac{53}{1120} - \frac{\sqrt{2}}{2520}\right) hf_{n+p_1} + \left(\frac{2}{21} + \frac{31\sqrt{2}}{672}\right) hf_{n+p_2} + \left(\frac{29}{1260} - \frac{22\sqrt{2}}{315}\right) hf_{n+p_3} + \left(\frac{17}{105} - \frac{41\sqrt{2}}{420}\right) hf_{n+p_4} + \left(\frac{11}{140} - \frac{22\sqrt{2}}{315}\right) hf_{n+p_5} + \left(\frac{2}{21} - \frac{13\sqrt{2}}{224}\right) hf_{n+p_6} + \left(-\frac{13}{10080} - \frac{\sqrt{2}}{2520}\right) hf_{n+p_7}\right)$$

$$y'_{n+p_3} = f(x_{n+p_3}, y_{n+p_3})$$

$$y'_{n+p_3} = f\left(x_{n+p_3}, y_n + \left(\frac{1829}{40320}\right) hf_{n+p_1} + \left(\frac{37}{336} + \frac{9\sqrt{2}}{128}\right) hf_{n+p_2} + \left(-\frac{41}{2520}\right) hf_{n+p_3} + \left(\frac{31}{1680}\right) hf_{n+p_4} + \left(-\frac{41}{2520}\right) hf_{n+p_5} + \left(\frac{37}{336} - \frac{9\sqrt{2}}{128}\right) hf_{n+p_6} + \left(-\frac{61}{40320}\right) hf_{n+p_7}\right)$$

$$y'_{n+p_4} = f(x_{n+p_4}, y_{n+p_4})$$

$$y'_{n+p_4} = f\left(x_{n+p_4}, y_n + \left(\frac{16}{315}\right) hf_{n+p_1} + \left(\frac{2}{21} + \frac{\sqrt{2}}{24}\right) hf_{n+p_2} + \left(\frac{17}{105}\right) hf_{n+p_3} + \left(\frac{17}{105}\right) hf_{n+p_4} + \left(-\frac{19}{315}\right) hf_{n+p_5} + \left(\frac{2}{21} - \frac{\sqrt{2}}{24}\right) hf_{n+p_6} + \left(\frac{1}{210}\right) hf_{n+p_7}\right)$$

$$y'_{n+p_5} = f(x_{n+p_5}, y_{n+p_5})$$

$$y'_{n+p_5} = f\left(x_{n+p_5}, y_n + \left(\frac{213}{4480}\right) hf_{n+p_1} + \left(\frac{9}{112} + \frac{9\sqrt{2}}{128}\right) hf_{n+p_2} + \left(\frac{33}{280}\right) hf_{n+p_3} + \left(\frac{171}{560}\right) hf_{n+p_4} + \left(\frac{33}{280}\right) hf_{n+p_5} + \left(\frac{9}{112} - \frac{9\sqrt{2}}{128}\right) hf_{n+p_6} + \left(\frac{3}{448}\right) hf_{n+p_7}\right)$$

$$y'_{n+p_6} = f(x_{n+p_6}, y_{n+p_6})$$

$$y'_{n+p_6} = f\left(x_{n+p_6}, y_n + \left(\frac{53}{1120} + \frac{\sqrt{2}}{2520}\right) hf_{n+p_1} + \left(\frac{2}{21} + \frac{13\sqrt{2}}{224}\right) hf_{n+p_2} + \left(\frac{29}{1260} + \frac{22\sqrt{2}}{315}\right) hf_{n+p_3} + \left(\frac{17}{105} + \frac{41\sqrt{2}}{420}\right) hf_{n+p_4} + \left(\frac{11}{140} + \frac{22\sqrt{2}}{315}\right) hf_{n+p_5} + \left(\frac{2}{21} - \frac{31\sqrt{2}}{672}\right) hf_{n+p_6} + \left(-\frac{13}{10080} + \frac{\sqrt{2}}{2520}\right) hf_{n+p_7}\right)$$

$$y'_{n+p_7} = f\left(x_{n+p_7}, y_n + \left(\frac{29}{630}\right) hf_{n+p_1} + \left(\frac{4}{21}\right) hf_{n+p_2} + \left(\frac{32}{315}\right) hf_{n+p_3} + \left(\frac{34}{105}\right) hf_{n+p_4} + \left(\frac{32}{315}\right) hf_{n+p_5} + \left(\frac{4}{21}\right) hf_{n+p_6} + \left(\frac{29}{630}\right) hf_{n+p_7}\right)$$

Putting

$$y'_{n+p_1} = f(x_{n+p_1}, y_{n+p_1}) = f_{n+p_1} = k_1, y'_{n+p_2} = f(x_{n+p_2}, y_{n+p_2}) = f_{n+p_2} = k_2,$$

$$y'_{n+p_3} = f(x_{n+p_3}, y_{n+p_3}) = f_{n+p_3} = k_3, y'_{n+p_4} = f(x_{n+p_4}, y_{n+p_4}) = f_{n+p_4} = k_4,$$

$$y'_{n+p_5} = f(x_{n+p_5}, y_{n+p_5}) = f_{n+p_5} = k_5, y'_{n+p_6} = f(x_{n+p_6}, y_{n+p_6}) = f_{n+p_6} = k_6,$$

$$y'_{n+p_7} = f(x_{n+p_7}, y_{n+p_7}) = f_{n+p_7} = k_7$$

We obtain the function evaluation as

$$k_1 = 0$$

$$k_2 = f \left( x_n + \left( \frac{1}{2} - \frac{\sqrt{2}}{4} \right) h, y_n + \left( \frac{53}{1120} - \frac{\sqrt{2}}{2520} \right) hf_{n+p_1} + \left( \frac{2}{21} + \frac{31\sqrt{2}}{672} \right) hf_{n+p_2} \right. \\ \left. + \left( \frac{29}{1260} - \frac{22\sqrt{2}}{315} \right) hf_{n+p_3} + \left( \frac{17}{105} - \frac{41\sqrt{2}}{420} \right) hf_{n+p_4} + \left( \frac{11}{140} - \frac{22\sqrt{2}}{315} \right) hf_{n+p_5} \right. \\ \left. + \left( \frac{2}{21} - \frac{13\sqrt{2}}{224} \right) hf_{n+p_6} + \left( -\frac{13}{10080} - \frac{\sqrt{2}}{2520} \right) hf_{n+p_7} \right)$$

$$k_3 = f \left( x_n + \left( \frac{1}{4} \right) h, y_n + \left( \frac{1829}{40320} \right) hf_{n+p_1} + \left( \frac{37}{336} + \frac{9\sqrt{2}}{128} \right) hf_{n+p_2} + \left( -\frac{41}{2520} \right) hf_{n+p_3} + \right. \\ \left. \left( \frac{31}{1680} \right) hf_{n+p_4} + \left( -\frac{41}{2520} \right) hf_{n+p_5} + \left( \frac{37}{336} - \frac{9\sqrt{2}}{128} \right) hf_{n+p_6} + \left( -\frac{61}{40320} \right) hf_{n+p_7} \right)$$

$$k_4 = f \left( x_n + \left( \frac{1}{2} \right) h, y_n + \left( \frac{16}{315} \right) hf_{n+p_1} + \left( \frac{2}{21} + \frac{\sqrt{2}}{24} \right) hf_{n+p_2} + \left( \frac{17}{105} \right) hf_{n+p_3} + \left( \frac{17}{105} \right) hf_{n+p_4} + \right. \\ \left. \left( -\frac{19}{315} \right) hf_{n+p_5} + \left( \frac{2}{21} - \frac{\sqrt{2}}{24} \right) hf_{n+p_6} + \left( \frac{1}{210} \right) hf_{n+p_7} \right)$$

$$k_5 = f \left( x_n + \left( \frac{3}{4} \right) h, y_n + \left( \frac{213}{4480} \right) hf_{n+p_1} + \left( \frac{9}{112} + \frac{9\sqrt{2}}{128} \right) hf_{n+p_2} + \left( \frac{33}{280} \right) hf_{n+p_3} + \left( \frac{171}{560} \right) hf_{n+p_4} + \right. \\ \left. \left( \frac{33}{280} \right) hf_{n+p_5} + \left( \frac{9}{112} - \frac{9\sqrt{2}}{128} \right) hf_{n+p_6} + \left( \frac{3}{448} \right) hf_{n+p_7} \right)$$

$$k_6 = f \left( x_n + \left( \frac{1}{2} + \frac{\sqrt{2}}{4} \right) h, y_n + \left( \frac{53}{1120} + \frac{\sqrt{2}}{2520} \right) hf_{n+p_1} + \left( \frac{2}{21} + \frac{13\sqrt{2}}{224} \right) hf_{n+p_2} \right. \\ \left. + \left( \frac{29}{1260} + \frac{22\sqrt{2}}{315} \right) hf_{n+p_3} + \left( \frac{17}{105} + \frac{41\sqrt{2}}{420} \right) hf_{n+p_4} + \left( \frac{11}{140} + \frac{22\sqrt{2}}{315} \right) hf_{n+p_5} \right. \\ \left. + \left( \frac{2}{21} - \frac{31\sqrt{2}}{672} \right) hf_{n+p_6} + \left( -\frac{13}{10080} + \frac{\sqrt{2}}{2520} \right) hf_{n+p_7} \right)$$

$$k_7 = f \left( x_n + (1)h, y_n + \left( \frac{29}{630} \right) hf_{n+p_1} + \left( \frac{4}{21} \right) hf_{n+p_2} + \left( \frac{32}{315} \right) hf_{n+p_3} + \left( \frac{34}{105} \right) hf_{n+p_4} + \right. \\ \left. \left( \frac{32}{315} \right) hf_{n+p_5} + \left( \frac{4}{21} \right) hf_{n+p_6} + \left( \frac{29}{630} \right) hf_{n+p_7} \right)$$

The weight  $b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ , evaluating the continuous scheme at  $x = x_n + h$ , we obtain

$$b = \left( \frac{29}{630}, \frac{120}{630}, \frac{64}{630}, \frac{204}{630}, \frac{64}{630}, \frac{120}{630}, \frac{29}{630} \right)$$

The seven point general formula for Runge-kutta method is defined as

$$y_{n+1} = y_n + h \sum_{j=1}^7 b_j k_j = y_n + \frac{29}{630} h(k_1 + k_7) + \frac{120}{630} h(k_2 + k_6) + \frac{64}{630} h(k_3 + k_5) + \frac{204}{630} h k_4$$

Where  $k_j, j = 1, 2, \dots, 7$  are given by (3.24), can be summarized as already show in table 1.1.

Table 1.1 Butcher table summary

C	A	U
	B	V

Where  $C = (c_1, c_2 \dots c_7)^T$ ,  $A = a_{ij}, ij = 1, 2 \dots 7$ ,  $U = (1, 1, 1, 1, 1, 1, 1)^T$ ,  $V = (1)$ ,

$B = (b_1, b_2 \dots b_7)^T$

Table 1.2 Butcher table summary of seven point cosine function Runge-kutta method

$C(P_i)$	$A = (a_{ij}), i, j = 1, 2, \dots, 7, A = (a_{ij})$						
0	0	0	0	0	0	0	0
$\frac{1}{2} - \frac{\sqrt{2}}{4}$	$(\frac{53}{1120} - \frac{\sqrt{2}}{2520})$	$(\frac{2}{21} + \frac{31\sqrt{2}}{672})$	$(\frac{29}{1260} - \frac{22\sqrt{2}}{315})$	$(\frac{17}{105} - \frac{41\sqrt{2}}{420})$	$(\frac{11}{140} - \frac{22\sqrt{2}}{315})$	$(\frac{2}{21} - \frac{13\sqrt{2}}{224})$	$(-\frac{13}{10080} - \frac{\sqrt{2}}{2520})$
$\frac{1}{4}$	$(\frac{1829}{40320})$	$(\frac{37}{336} + \frac{9\sqrt{2}}{128})$	$(-\frac{41}{2520})$	$(\frac{31}{1680})$	$(-\frac{41}{2520})$	$(\frac{37}{336} - \frac{9\sqrt{2}}{128})$	$(-\frac{61}{40320})$
$\frac{1}{2}$	$(\frac{16}{315})$	$(\frac{2}{21} + \frac{\sqrt{2}}{24})$	$(\frac{17}{105})$	$(\frac{17}{105})$	$(-\frac{19}{315})$	$(\frac{2}{21} - \frac{\sqrt{2}}{24})$	$(\frac{1}{210})$
$\frac{3}{4}$	$(\frac{213}{4480})$	$(\frac{9}{112} + \frac{9\sqrt{2}}{128})$	$(\frac{33}{280})$	$(\frac{171}{560})$	$(\frac{33}{280})$	$(\frac{9}{112} - \frac{9\sqrt{2}}{128})$	$(\frac{3}{448})$
$\frac{1}{2} + \frac{\sqrt{2}}{4}$	$(\frac{53}{1120} + \frac{\sqrt{2}}{2520})$	$(\frac{2}{21} + \frac{13\sqrt{2}}{224})$	$(\frac{29}{1260} + \frac{22\sqrt{2}}{315})$	$(\frac{17}{105} + \frac{41\sqrt{2}}{420})$	$(\frac{11}{140} + \frac{22\sqrt{2}}{315})$	$(\frac{2}{21} - \frac{31\sqrt{2}}{672})$	$(-\frac{13}{10080} + \frac{\sqrt{2}}{2520})$
1	$(\frac{29}{630})$	$(\frac{4}{21})$	$(\frac{32}{315})$	$(\frac{34}{105})$	$(\frac{32}{315})$	$(\frac{4}{21})$	$(\frac{29}{630})$
$b$	$\frac{29}{630}$	$\frac{120}{630}$	$\frac{64}{630}$	$\frac{204}{630}$	$\frac{64}{630}$	$\frac{120}{630}$	$\frac{29}{630}$

### ANALYSIS OF THE METHOD

1. Consistency: the Runge-Kutta method is consistent since

$$\sum_{j=1}^7 a_{ij} = c_i, \sum_{j=1}^7 b_i = 1, \text{ (see Table 1.2)}$$

2. Stability: the stability of the method is investigated by considering the linear test equation.

$$y' = \lambda y, \quad \lambda \in \mathbb{C}$$

Putting  $Z = \lambda h, \quad h \in (0,1)$

The stability function is  $R(Z)$

$$R(Z) = I + Zb^T(I - ZA)^{-1}e$$

Where  $I$  is the identity matrix,  $b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$  is the weight,  $e = (1,1,1,1,1,1,1)$ ,  $A$  is the Runge-Kutta matrix of the coefficient of the butcher.

**3. Order and error constant of the method:** the Runge-Kutta solution  $y_{n+1}$  can be expanded into Taylor's series as

$$y_{n+1} = y_n + h \sum_{i=1}^n b_i k_i = y_n + h \sum_{i=1}^n b_i f(x_{n+p_i}, y_{n+p_i}) = y_n + h \sum_{i=1}^n b_i y'_{n+p_i}$$

$$y_{n+1} = (y_n + h \sum_{i=1}^n b_i y'_{n+p_i})$$

$$y_{n+1} = (y_n + b_1 h y'_{n+p_1} + b_2 h y''_{n+p_2} + b_3 h y'''_{n+p_3} + b_4 h y^4_{n+p_4} \dots b_n h y^n_{n+p_n})$$

Where  $p_i$  are the special points from the five points of cosine function and the weight

$$b = (b_1, b_2, b_3, \dots, b_n).$$

The Taylor expansions of  $y(x_{n+1}) - y_{n+1}$

The order of our scheme is  $p = 8$ , since  $c_0 = c_1 \dots = c_8 = 0$  and  $c_9 = -\frac{1}{1857945600}$ , error constant. The Runge-Kutta solution agrees with Taylor's series expansion up to the term in  $h^8$ . The truncation error is  $o(h^9)$ .

### NUMERICAL EXPERIMENTS/ RESULTS

To show the efficiency of this new method, we use two problems to compare our computational solution with exact and some existing method of the same order.

Problem 1:  $y' = -50y + 50 \sin(t), y(0) = 1, h = 0.1$ , Analytical solution:  $y(t) = -\frac{50}{2501} \cos(t) + \frac{2500}{2501} \sin(t) + \frac{2551}{2501} e^{-50t}$

Problem 2:  $y' = \frac{y}{4} \left(1 - \frac{y}{20}\right), y(0) = 1, h = 0.1$ , Analytical solution:  $y(t) = \frac{20}{1+19e^{-\frac{t}{4}}}$

The following are solution tables for the given problems above.

Table 4.2: Comparison of numerical solution of problem 1

t	EXACT SOLUTION	NEW METHOD OF 7 POINT (CF67)	YAKUBU&BABU ERROR	NEW METHOD OF 7 POINT(CF67) ERROR
0.1	0.0867740248491948	0.086897355284545	$2.874 \times 10^{-3}$	$1.233 \times 10^{-4}$
0.2	0.179042708243269	0.179044385143413	$3.063 \times 10^{-5}$	$1.677 \times 10^{-6}$
0.3	0.276303267714471	0.276303284815288	$2.506 \times 10^{-7}$	$1.710 \times 10^{-8}$
0.4	0.370848785017787	0.370848785172806	$1.862 \times 10^{-9}$	$1.550 \times 10^{-10}$
0.5	0.461689211695889	0.461689211697206	$1.323 \times 10^{-11}$	$1.317 \times 10^{-12}$
0.6	0.547916594459153	0.547916594459166	$1.31 \times 10^{-13}$	$1.317 \times 10^{-14}$
0.7	0.628669375741705	0.628669375741706	$4.0 \times 10^{-14}$	$1.0 \times 10^{-15}$
0.8	0.703140700432406	0.703140700432407	$3.0 \times 10^{-14}$	$1.0 \times 10^{-15}$
0.9	0.770586475671801	0.770586475671801	$2.9 \times 10^{-14}$	0
1.0	0.830332805568306	0.830332805568308	$2.3 \times 10^{-14}$	$1.0 \times 10^{-15}$

Table 4.1: Comparison of numerical solution of problem2 with YAKUBU &BABU (2011)

t	y(t)	NEW METHOD OF 7 POINT (CF7)	YAKUBU&BABU ERROR	NEW METHOD OF 7 POINT (CF7) ERROR
0.1	1.024018962351867	1.024018962351867	$5.984 \times 10^{-11}$	0
0.2	1.048582996382734	1.048582996382734	$1.219 \times 10^{-10}$	0
0.3	1.073702928838883	1.073702928838884	$1.864 \times 10^{-10}$	$1.0 \times 10^{-15}$
0.4	1.099389726731483	1.099389726731484	$2.534 \times 10^{-10}$	$1.0 \times 10^{-15}$
0.5	1.125654495329783	1.125654495329783	$3.229 \times 10^{-10}$	0

## DISCUSSION/ CONCLUSION

We used three test problem. We see that the error of the new method is less than that of Yakubu & Babu (2011). We have been able to construct new Runge-Kutta method using cosine functions. This method is easier than Yakubu method (2011) already existing.

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