BAYESIAN AND CLASSICAL ESTIMATION OF PARAMETER AND RELIABILITY OF WEIBEXPO DISTRIBUTION

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ABSTRACT

This paper deals with the Bayesian and classical estimation of parameter θ of a distribution known as the Weibexpo Distribution .The estimation has been performed for type II censored samples. In classical setup the Maximum Likelihood Estimator (MLE) and the Unique Minimum Variance Unbiased (UMVU) Estimator of θ and the Reliability of the distribution have been obtained. In the Bayesian setup estimates of θ and the reliability of the distribution have been obtained .The estimation has been performed by taking a Natural Conjugate prior distribution for θ and four different types of loss functions. On the part of loss functions, the Squared Error Loss Function (SELF), DeGroot Loss Function (DLF), Minimum Expected Loss (MELO) Function and Exponentially Weighted Minimum Expected Loss (EWMELO) Function have been considered. Bayes Risks of Bayes estimators corresponding to four loss functions have also been obtained.

Keywords: Weibexpo Distribution, Maximum Likelihood Estimator, Unique Minimum Variance Unbiased Estimator, Reliability, Type II Censoring Bayes Estimator, Squared Error Loss Function (SELF), DeGroot Loss Function (DLF), Minimum Expected Loss (MELO) Function and Exponentially Weighted Minimum Expected Loss (EWMELO) Function.Bayes Risk.

1.INTRODUCTION

Consider a system consisting of $(p+q)$ components connected in series. Where,$p \geq 0, q \geq 0$,.Let X_i be the life of the i^{th} component ,$i = 1,2 \dots p+q$.For $i = 1,2 \dots . p$,let the distribution of X_i has exponential distribution with common probability density function given by,

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, \text{if } x > 0, \theta > 0 \\ 0, \text{Otherwise} \end{cases} \quad (1.1)$$

In this case, $P(X_i > t) = e^{-\frac{t}{\theta}}$, $i = 1,2 \dots . p$

While for the remaining q components distribution of X_i has Weibull distribution with common probability density function given by,

$$g(x,\alpha,\theta) = \begin{cases} \frac{\alpha x^{\alpha-1}}{\theta} e^{-\frac{x^\alpha}{\theta}}, \text{if } x > 0, \theta > 0, \alpha > 0 \\ 0, \text{Otherwise} \end{cases} \quad (1.2)$$

In this case, $P(X_i > t) = e^{-\frac{t^\alpha}{\theta}}$, $i = p+1, p+2 \dots . p+q$

Let Y be the life of the system. Since the components are connected in series, $Y =$ Mininmum (X_1 , $X_2, \dots X_{p+q}$).The reliability of the system, denoted by $R(t) = P(Y > t)$,is given by,

$$R(t,\alpha,\theta) = P(Y > t) = \prod_{i=1}^{p+q} P(X_i > t) = e^{-\frac{(pt+qt^\alpha)}{\theta}} \quad (1.3)$$

The probability density function of Y is given by, $h(y,\alpha,\theta) = -R'(y,\alpha,\theta)$ and so,

$$h(y,\alpha,\theta) = \begin{cases} \frac{(p+q\alpha y^{\alpha-1})}{\theta} e^{-\frac{(py+qy^\alpha)}{\theta}}, \text{if } y > 0, \theta > 0, \alpha > 0 \\ 0, \text{Otherwise} \end{cases} \quad (1.4)$$

For $p = 1$ and $q = 0$,(1.4) reduces to exponential distribution with parameter θ while for $p = 0$ and $q = 1$,(1.4) reduces to Weibull distribution with shape parameter α and scale parameter θ. Remark :1.It is to be noted that (1.4) always represents a probability density function even if $p > 0$, $q > 0$ are not necessarily positive integers. For $p = 0$,$q = 1$ and $\alpha = 2$,(1.4) reduces to Rayleigh distribution

Remark :2.It is to be noted that the random variable $U = (pY + qY^\alpha)$ has the same probability density function given by (1.1) and so,

$$h_1(u,\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{u}{\theta}}, \text{if } u > 0, \theta > 0 \\ 0, \text{Otherwise} \end{cases} \quad (1.5)$$

We call the distribution having p. d. f given by (1.4) as Weibexpo Distribution.

In this paper, classical and Bayesian procedure has been adopted to obtain estimates of θ and $R(t,\alpha,\theta)$,when α is known. The estimation has been performed under type II censoring. The Maximum Likelihood Estimator (M.L.E.) and Unique Minimum Variance Unbiased Estimator (UMVUE) have been obtained. In Bayesian framework, estimation has been performed under the assumption of a Natural Conjugate Prior density for θ. The loss functions considered are as under:

1.The Squared Error Loss Function (SELF): In this case, the loss function denoted by $L(\theta,\delta)$, is given by,

$$L(\theta,\delta) = (\theta-\delta)^2 \quad (1.6)$$

This loss function is symmetric and unbounded. It suffers from the drawback of giving equal weights to underestimation as well as to overestimation.

2. DeGroot Loss Function (DLF): In this case

$$L(\theta,\delta) = \delta^{-2}(\theta-\delta)^2 \quad (1.7)$$

This loss function, introduced by DeGroot (2005), is asymmetric. It gives more weight to underestimation than to overestimation.

3.Minimum Expected Loss (MELO) Function: In this case,

$$L(\theta,\delta) = \theta^{-2}(\theta-\delta)^2 \quad (1.8)$$

This loss function is asymmetric and bounded. In this case weight due to underestimation and overestimation is changed by a factor θ^{-2} as compared to the SELF. This loss function was used by Tummala and Sathe (1978) for estimating reliability of certain life time distribution and by Zellner (1979) for estimating functions of parameters in econometric models.

4. Exponentially Weighted Minimum Expected Loss (EWMELO) Function

$$w(\theta,\delta) = \theta^{-2} e^{-a\theta^{-1}}(\theta-\delta)^2 \quad (1.9)$$

This loss function is asymmetric and bounded. In this case weight due to underestimation and overestimation is changed by a factor $e^{-a\theta^{-1}}$ as compared to the MELO and by a factor $\theta^{-2}e^{-a\theta^{-1}}$ as compared to the SELF. This type of loss function was used by the author (1997) for the first time in his work for D.Phil. SELF, MELO and EWMELO were used by Singh, the author, (1999) in the study of reliability of a multicomponent system and (2010) in Bayesian Estimation of the mean and distribution function of Maxwell's distribution. Recently, the author again used these loss functions in Bayesian estimation of function of the unknown parameter θ for the Modified Power Series Distribution (MPSD) (2021) ,for estimating Loss and Risk Functions of a continuous distribution (2021),for estimating moments and reliability of Geometric distribution. In addition to these loss functions, the author has used Degroot loss function while estimating the unknown parameter and reliability of Burr Type XII distribution.

The results obtained in this paper give classical and Bayesian estimates of unknown parameter θ for a number of distributions as particular cases as under:

For $p = 1$ and $q = 0$ results obtained for the MLE and UMVUE coincide with that given by Epstein and Sobel (1953) for the exponential distribution while under SELF the Bayes estimates coincide with that given by Bhattacharya (1967) corresponding to natural conjugate prior.Under MELO we get the result for exponential distribution similar to that as given by Tummala and Sathe (1978). For $p = 0$,$q = 1$ and known value of α ,we get results for MLE and UMVUE of Weibull distribution while under MELO the Bayes estimates are similar to that given by Tummala and Sathe (1978).For $p = 0$,$q = 1$ and $\alpha = 2$ results obtained for the MLE and UMVUE coincide with that of Rayleigh distribution while under SELF the Bayes estimates are similar to that given by Bhattacharya and Tyagi (1990) for the natural conjugate prior.

2.ESTIMATION OF θ AND $R(t,\alpha,\theta)$ UNDER type II CENSORING

Let $Y_1, Y_2, Y_3,\ldots Y_n$ be a random sample of size n and $Y_{(1)} < Y_{(2)} < Y_{(3)} <\ldots< Y_{(n-1)} < Y_{(n)}$ be the order statistic corresponding to this random sample having p.d.f given by (1.4). In case of type II censoring, n items are placed on test and the test is terminated after first 'r' (r pre-specified) failures. Thus, only r ordered observed values of $Y_{(1)} < Y_{(2)} < Y_{(3)} <\ldots< Y_{(r-1)} < Y_{(r)}$ are recorded. For observed values $y_{(1)} < y_{(2)} < y_{(3)} <\ldots< y_{(r-1)} < y_{(r)}$,the likelihood function, when α is known ,denoted by $L(\theta)$,is given by,

$L(\theta) = k\theta^{-r}e^{-\frac{t_r}{\theta}}$ (2.1)

Where, k is function of n, p, q, r and $y_{(i)}$, i=1,2…r and does not contain θ.

$t_r = \sum_{i=1}^{r} u_{(i)} + (n-r)u_{(r)}$ (2.2)

Where,$u_{(i)} = py_{(i)} + qy_{(i)}^{\alpha}$,i=1,2…r.

t_r is an observed value of the statistic T_r given by,

$T_r = \sum_{i=1}^{r} U_{(i)} + (n-r)U_{(r)}$

Where,$U_{(i)} = pY_{(i)} + qY_{(i)}^{\alpha}$,i=1,2…r (2.3)

$$l(\theta) = \ln L(\theta) = -r\ln\theta - \frac{t_r}{\theta}$$

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{-r}{\theta} + \frac{t_r}{\theta^2}$$

Solving $\frac{\partial l(\theta)}{\partial\theta} = 0$ and denoting the solution by $\hat{\theta}$, we have,

$\hat{\theta} = \frac{t_r}{r}$ (2.4)

Thus, the Maximum Likelihood Estimate of θ is $\hat{\theta} = \frac{t_r}{r}$.The corresponding estimator is, $\frac{T_r}{r}$.

The estimator $\frac{T_r}{r}$ is also UMVU and efficient for θ.This is proved as under:

The p. d. f of $V = \frac{T_r}{r}$ is derived and is given by,

$$g_2(v,\theta) = \begin{cases} \frac{1}{\Gamma(r)}(\frac{r}{\theta})^r v^{r-1} e^{-\frac{rv}{\theta}}, \text{if } v > 0, \theta > 0 \\ 0, otherwise. \end{cases} \quad (2.5)$$

PROOF

If $Y_{(1)} < Y_{(2)} < Y_{(3)} <\ldots< Y_{(n-1)} < Y_{(n)}$ is the order statistic corresponding to a random sample of size n having p.d.f given by (1.4) then, $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$ constitute an order statistic of size n corresponding to a random sample of size n from the distribution specified by (1.5),where, $U_{(i)} = pY_{(i)} + qY_{(i)}^{\alpha}$.Since $U = (pY + qY^{\alpha})$ has the p.d,f. given by (1.5).It is to be noted that $P(Y_{(1)} > t) = e^{-\frac{n(pt+qt^{\alpha})}{\theta}} = P(U_{(1)} > pt + qt^{\alpha})$.The joint p.d.f. of , $U_{(1)} < U_{(2)} < \cdots < U_{(r)}$,is given by,

$$h_*\left(u_{(1)}, u_{(2)}, u_{(3)}, \ldots, u_{(r-1)}, u_{(r)}, \theta\right)$$
$$= \frac{n!}{(n-r)!}\{\prod_{i=1}^{r} h_1(u_{(i)},\theta)\}\{R\left(u_{(r)},\theta\right)\}^{(n-r)} , if\ 0 < u_{(1)} < u_{(2)} < u_{(3)} < \cdots < u_{(r-1)} < u_{(r)} < \infty; \theta > 0$$
$$= \{\prod_{i=1}^{r}(\frac{(n-i+1)}{\theta})\}\ e^{-\frac{t_r}{\theta}}$$

Let $L_1 = U_{(1)} = pY_{(1)} + qY_{(1)}^{\alpha}$,

$L_i = U_{(i)} - U_{(i-1)}$, i=2,3,4…..r. Then,

$\sum_{i=1}^{r}(n-i+1)L_i = \sum_{i=1}^{r} U_{(i)} + (n-r)U_{(r)} = T_r$.

The Jacobian of the transformation from $(U_{(1)}, U_{(2)}, U_{(3)}, \ldots, U_{(r-1)}, U_{(r)})$ to $(L_1, L_2, \ldots. L_r)$ is1.So the joint p. d. f. of $L_1, L_2, \ldots. L_r$ say , $g_1(l_1, l_2, \ldots. l_r)$,is given by,

$$g_1(l_1, l_2, \ldots. l_r) = \begin{cases} \prod_{i=1}^{r}\left(\frac{(n-i+1)}{\theta}\right)\ e^{-\frac{\sum_{i=1}^{r}(n-i+1)l_i}{\theta}}, 0 < l_1 < l_2 < l_3 < \cdots < l_{r-1} < l_r < \infty \\ 0, otherwise \end{cases}$$

This shows that $L_1, L_2, \ldots. L_r$are mutually independent and for each $i, V_i = (n-i+1)L_i$ has exponential distribution having p. d. f.

$$g_i(v_i,\theta) = \begin{cases} \frac{1}{\theta}e^{-\frac{v_i}{\theta}}, if\ v_i > 0, \theta > 0 \\ 0, otherwise \end{cases}$$

Hence, $V = \frac{T_r}{r} = \frac{\sum_{i=1}^{r}(n-i+1)L_i}{r} = \frac{\sum_{i=1}^{r} V_i}{r}$,has p. d. f.

$g_2(v,\theta) = \begin{cases} \frac{1}{\Gamma(r)}\left(\frac{r}{\theta}\right)^r v^{r-1} e^{-\frac{rv}{\theta}}, \text{if } v > 0, \theta > 0. \\ 0, otherwise. \end{cases}$

$E(V) = E\left(\frac{T_r}{r}\right) = \int_0^\infty v\, g_2(v,\theta) dv = \theta$.

This shows that $\frac{T_r}{r}$ is an unbiased estimator of θ.

Also, Var($\frac{T_r}{r}$) = Var(V)= $\int_0^\infty v^2\, g_2(v,\theta)dv - \theta^2 = \frac{\theta^2}{r}$

Since, the joint p. d. f. of $Y_{(1)} < Y_{(2)} < Y_{(3)} < \ldots < Y_{(r-1)} < Y_{(r)}$ is given by,

$f_*\big(y_{(1)}, y_{(2)}, y_{(3)}, \ldots, y_{(r-1)}, y_{(r)}, \theta\big) = \{\prod_{i=1}^{r}(\frac{(n-i+1)(p+\alpha q y_{(i)})}{\theta})\}\ e^{-\frac{t_r}{\theta}}$

$k_1(t_r,\theta).\, k_2\big(y_{(1)}, y_{(2)}, y_{(3)}, \ldots, y_{(r-1)}, y_{(r)}\big)$

Where, $k_1(t_r,\theta) = \theta^{-r} e^{-\frac{t_r}{\theta}}$,

$k_2\big(y_{(1)}, y_{(2)}, y_{(3)}, \ldots, y_{(r-1)}, y_{(r)}\big) = \{\prod_{i=1}^{r}(\frac{(n-i+1)(p+\alpha q y_{(i)})}{\theta})\}$

Hence, by virtue of factorization criterion, T_r is sufficient statistic for θ. Since, $T_r \sim G(r,\theta)$ and the family of Gamma distribution, with known parameter r and unknown parameter θ ,is complete, T_r is complete sufficient statistic for θ.Hence,by virtue of Rao-Blackwell -Lehman-Scheff$\acute{e}$ Theorem, $\frac{T_r}{r}$,is Unique Minimum Variance Unbiased(UMVU) estimator of θ.

Since,

$$\frac{\partial ln f_*\big(y_{(1)}, y_{(2)}, y_{(3)}, \ldots, y_{(r-1)}, y_{(r)}, \theta\big)}{\partial\theta} = -\frac{r}{\theta} + \frac{r}{\theta^2}\frac{t_r}{r}$$

$$\frac{\partial^2 ln f_*\big(y_{(1)}, y_{(2)}, y_{(3)}, \ldots, y_{(r-1)}, y_{(r)}, \theta\big)}{\partial\theta^2} = \frac{r}{\theta^2} - \frac{2r}{\theta^3}\frac{t_r}{r}$$

$-E\left\{\frac{\partial^2 ln f_*\left(Y_{(1)}, Y_{(2)}, Y_{(3)}, \ldots, Y_{(r-1)}, Y_{(r)}, \theta\right)}{\partial\theta^2}\right\} = -\frac{r}{\theta^2} + \frac{2r}{\theta^3}E(\frac{T_r}{r}) = \frac{r}{\theta^2}$.

Hence, the Cram$\acute{e}$r-Rao Lower bound = $\frac{1}{-E\left\{\frac{\partial^2 ln f_*\left(Y_{(1)}, Y_{(2)}, Y_{(3)}, \ldots, Y_{(r-1)}, Y_{(r)}, \theta\right)}{\partial\theta^2}\right\}} = \frac{\theta^2}{r}$

Since,$Var(\frac{T_r}{r}) = \frac{\theta^2}{r}$.So the Variance of $\frac{T_r}{r}$ attains the Cram$\acute{e}$r-Rao Lower bound and therefore, $\frac{T_r}{r}$ is an efficient estimator of θ.

Consider random variables,

$Q_1, Q_2, \ldots. Q_r$ as follows: $Q_i = \frac{V_i}{\sum_{i=1}^{r} V_i}$, for $i = 1,2 \ldots. r-1$

and $Q_r = \sum_{i=1}^{r} V_i$

The transformations $q_i = \frac{v_i}{\sum_{i=1}^{r} v_i}$ and $q_r = \sum_{i=1}^{r} v_i$

map the set $\mathcal{A} = \{(v_1, v_2, \ldots. v_r): v_i > 0, i = 1,2 \ldots r\}$ onto the set $\mathcal{B} = \{(q_1, q_2, \ldots. q_r): 0 \le q_i, i = 1,2, \ldots r-1; 0 \le q_r < \infty, q_1 + q_2 + \cdots + q_{r-1} \le 1\}$.The inverse transformations are,

$v_i = q_i q_r, i = 1,2..r-1$ and $v_r = q_r(1 - q_1 - q_2 - \cdots - q_{r-1})$.The Jacobian of this transformation, denoted by J ,is given by,

$$J = \frac{\partial(v_1, v_2, \ldots v_r)}{\partial(q_1, q_2, \ldots. q_r)} = \begin{vmatrix} q_r\ 0 \ldots. q_1 \\ 0\ q_r \ldots. q_2 \\ 0\ 0\ q_r..q_3 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ -q_r - q_r..b \end{vmatrix}, b = (1 - q_1 - q_2 - \cdots - q_{r-1}).$$

Or.

$J = q_r^{r-1}$

The joint p. d. f. of $Q_1, Q_2, \ldots. Q_r$, denoted by$g_*\big(q_1, q_2, \ldots. q_{r,}\theta\big)$

$g_*\left(q_1, q_2, \ldots . q_{r,} \theta\right)=\begin{cases}\theta^{-r} u_r^{r-1} e^{-\frac{u_r}{\theta}}, if\left(q_1, q_2, \ldots . q_r\right) \in \mathcal{B} \\ 0, otherwise\end{cases}$

Or,

$g_*\left(q_1, q_2, \ldots . q_{r,} \theta\right)=\begin{cases}\frac{(r-1)!}{\Gamma(r)} \theta^{-r} q_r^{r-1} e^{-\frac{u_r}{\theta}}, if\left(q_1, q_2, \ldots . q_r\right) \in \mathcal{B} \\ 0, otherwise\end{cases}$

This shows that $(Q_1, Q_2, \ldots . Q_{r-1})$ is independent of Q_r and Q_r has p. d. f.

$g_{*1}\left(q_{r,} \theta\right)=\begin{cases}\frac{1}{\Gamma(r)} \theta^{-r} q_r^{r-1} e^{-\frac{q_r}{\theta}}, if\ 0 \leq q_r<\infty \\ 0, otherwise\end{cases}$

The random vector $(Q_1, Q_2, \ldots . Q_{r-1})$ has the p. d. f. of Dirichlet distribution and is given by,

$g_{2*}\left(q_1, q_2, \ldots . q_{r-1,}\right)=\begin{cases}(r-1)!, if 0 \leq q_i, i=1,2, \ldots r-1 ; q_1+q_2+\cdots+q_{r-1} \leq 1 \\ 0, otherwise\end{cases}$

The random vector $(Q_1, Q_2, \ldots . Q_{r-2})$ has the p. d. f. of Dirichlet distribution and is given by,

$$g_{3*}\left(q_1, q_2, \ldots . q_{r-2,}\right)=(r-1)! \int_0^{1-q_1-q_2-q_3-\cdots q_{r-2}} d q_{r-1}$$
$$=(r-1)!\,(1-q_1-q_2-q_3-\cdots q_{r-2})$$

Similarly,

The random vector $(Q_1, Q_2, \ldots . Q_{r-3})$ has the p. d. f. of Dirichlet distribution and is given by,

$g_{4*}(q_1, q_2, \ldots . q_{r-3})=(r-1)! \int_0^{1-q_1-q_2-q_3-\cdots q_{r-3}}(1-q_1-q_2-q_3-\cdots q_{r-3}) d q_{r-2}$
$=\frac{(r-1)!}{2}(1-q_1-q_2-q_3-\cdots q_{r-2})^2$

Proceeding this way ,we get the p. d. f of the random variable Q_1 ,as follows:

$g_{1*}(q_1)=\begin{cases}(r-1)(1-q_1)^{r-2}, if\ 0 \leq q_1 \leq 1 \\ 0, otherwise\end{cases}$

This shows that Q_1 has Beta $(1, r-1)$ distribution. Since, $the\ random\ vector$ $(Q_1, Q_2, \ldots . Q_{r-1})$ is independent of Q_r. In particular $Q_r=\sum_{i=1}^r V_i$ is independent of $Q_1=\frac{V_1}{\sum_{i=1}^r V_i}$

Since, $V_1=n L_1=n U_{(1)}$has p .d . f.

$$g_1(v_1, \theta)=\begin{cases}\frac{1}{\theta} e^{-\frac{v_1}{\theta}}, if\ v_1>0, \theta>0 \\ 0, otherwise\end{cases}$$

$$P(V_1>p t+q t^\alpha)=P\{(n U_{(1)}>p t+q t^\alpha)\}=e^{-\frac{(p t+q t^\alpha)}{\theta}}=R(t, \theta)$$

$Consider\ the\ statistic\ V_*$ defined as follows:

$$V_*=\begin{cases}1, if\ V_1>p t+q t^\alpha \\ 0, otherwise\end{cases}$$

$E(V_*)=1. P(V_1>p t+q t^\alpha))+0. P(V_1<p t+q t^\alpha)=e^{-\frac{(p t+q t^\alpha)}{\theta}}=R(t, \theta)$.

So V_* is an unbiased estimator of $R(t, \theta)$.By virtue of Rao-Blackwell-Lehman-Scheffé theorem,$\varphi(V_*)= E(V_* / \sum_{i=1}^r V_i)$ is UMVU estimator of $R(t, \theta)=e^{-\frac{(p t+q t^\alpha)}{\theta}}$.

Now, $\varphi(V_*)=E(V_* / \sum_{i=1}^r V_i)=P(V_1>p t+q t^\alpha / \sum_{i=1}^r V_i)=P\left(\frac{V_1}{\sum_{i=1}^r V_i}>\frac{p t+q t^\alpha}{\sum_{i=1}^r V_I} / \sum_{i=1}^r V_i\right)=P\left(Q_1>\frac{p t+q t^\alpha}{\sum_{i=1}^r V_I} / \sum_{i=1}^r V_i\right)=(r-1) \int_{\frac{p t+q t^2}{\sum_{i=1}^r V_i}}^1(1-q_1)^{r-2} d q_1=(1-\frac{p t+q t^\alpha}{\sum_{i=1}^r V_i})^{r-1}$ .Thus ,the UMVU estimator of $R(t, \theta)= e^{-\frac{(p t+q t^\alpha)}{\theta}}$,denoted by , $\hat{R}(t, \theta)$ is given by ,

$\hat{R}(t, \theta)=\begin{cases}(1-\frac{p t+q t^\alpha}{\sum_{i=1}^r V_i})^{r-1}, if\ p t+q t^\alpha<\sum_{i=1}^r V_i \\ 0, if\ p t+q t^\alpha \geq \sum_{i=1}^r V_i\end{cases}$

Since, $\sum_{i=1}^r V_i=T_r=\sum_{i=1}^r U_{(i)}+(n-r) U_{(r)}$

$$\hat{R}(t,\theta) = \begin{cases} (1 - \frac{pt + qt^{\alpha}}{T_r})^{r-1}, if\ pt + qt^{\alpha} < T_r \\ 0, if\ pt + qt^{\alpha} \geq T_r \end{cases}$$

3.BAYESIAN ESTIMATION

The Bayesian estimation of θ and $R(t,\theta)$ has been obtained by taking the Inverted gamma distribution as the prior probability distribution for θ and three different types of loss functions. The probability density function of the prior distribution for θ,denoted by $\pi(\theta)$,is given as follows:

$$\pi(\theta)=\begin{cases} \frac{\mu^{\nu}}{\Gamma(\nu)} \theta^{-(\nu+1)} e^{-\frac{\mu}{\theta}}, if\ 0 < \theta < \infty, \mu, \nu > 0 \\ 0, otherwise \end{cases} \quad (3.1)$$

Where, $\mu\ and\ \nu$ are known.

Since, in case of type II censoring, the likelihood function, denoted by $L(\theta)$,is given by,

$$L(\theta) = k\theta^{-r} e^{-\frac{t_r}{\theta}} \quad (3.2)$$

Where, k is function of n, p, q, r and $y_{(i)}$,i=1,2…r and does not contain θ.

The posterior distribution of θ,denoted by $\pi(\theta\ /t_r)$,is given by.

$$\pi(\theta\ /t_r) = \frac{L(\theta)\pi(\theta)}{\int_0^{\infty} L(\theta)\pi(\theta)d\theta}$$

$$= \frac{\theta^{-(\nu+r+1)} e^{-\frac{(t_r+\mu)}{\theta}}}{\int_0^{\infty} \theta^{-(\nu+r+1)} e^{-\frac{(t_r+\mu)}{\theta}} d\theta} = \frac{(t_r+\mu)^{r+\nu}}{\Gamma(r+\nu)} \theta^{-(\nu+r+1)} e^{-\frac{(t_r+\mu)}{\theta}}$$

Thus,

$$\pi(\theta\ /t_r)=\begin{cases} \frac{(t_r+\mu)^{r+\nu}}{\Gamma(r+\nu)} \theta^{-(\nu+r+1)} e^{-\frac{(t_r+\mu)}{\theta}}, if\ 0 < \theta < \infty, \mu, \nu > 0 \\ 0, otherwise\ . \end{cases}$$

It is to be noted that the posterior distribution $\pi(\theta\ /t_r)$ is also the probability density function of an Inverted gamma distribution. So $\pi(\theta)$ is a Natural Conjugate prior probability density for θ.

1.Under the Squared Error Loss Function given by, $L(\theta,\delta) = (\theta - \delta)^2$, the Bayes estimate of θ,denoted by $\hat{\theta}_B$, is given by,

$$\hat{\theta}_B = E(\theta\ /t_r) = \int_0^{\infty} \theta\pi(\theta\ /t_r)d\theta = \frac{(t_r+\mu)^{r+\nu}}{\Gamma(r+\nu)} \int_0^{\infty} \theta^{-(\nu+r)} e^{-\frac{(t_r+\mu)}{\theta}} d\theta = \frac{(t_r+\mu)}{(r+\nu-1)}$$

So,

$$\hat{\theta}_B = \frac{(t_r+\mu)}{(r+\nu-1)} \text{ ,provided, } r + \nu > 1 \quad (3.4)$$

2.Under the DeGroot Loss Function given by ,$L(\theta,\delta) = \delta^{-2}(\theta - \delta)^2$, the Bayes estimate of θ,denoted by $\hat{\theta}_{DG}$, is given by,

$$\hat{\theta}_{DG} = \frac{E(\theta^2\ /t_r)}{E(\theta\ /t_r)} = \frac{\int_0^{\infty} \theta^2\pi(\theta\ /t_r)d\theta}{\int_0^{\infty} \theta\pi(\theta\ /t_r)d\theta} = \frac{(t_r+\mu)}{(r+\nu-2)} \text{,provided, } r + \nu > 2 \quad (3.5)$$

3.Under the Minimum Expected Loss (MELO) Function, given by, $L(\theta,\delta) = \theta^{-2}(\theta - \delta)^2$,the Bayes estimate of θ,(also known as the Minimum Expected Loss (MELO) Estimate, denoted by $\hat{\theta}_M$, is given by,

$$\hat{\theta}_M = \frac{E(\theta^{-1}\ /t_r)}{E(\theta^{-2}\ /t_r)} = \frac{\int_0^{\infty} \theta^{-1}\pi(\theta\ /t_r)d\theta}{\int_0^{\infty} \theta^{-2}\pi(\theta\ /t_r)d\theta} = \frac{(t_r+\mu)}{(r+\nu+1)}$$

$$\hat{\theta}_M = \frac{(t_r+\mu)}{(r+\nu+1)} \quad (3.6)$$

4.Under the Exponentially Weighted Minimum Expected Loss (EWMELO) Function, given by, $L(\theta,\delta) = \theta^{-2}e^{-a\theta^{-1}}(\theta - \delta)^2$, the Bayes estimate of θ ,known as the Exponentially Weighted Minimum Expected Loss (MELO) Estimate, denoted by $\hat{\theta}_{EW}$, is given by,

$$\hat{\theta}_{EW} = \frac{E(\theta^{-1}e^{-a\theta^{-1}}\ /t_r)}{E(\theta^{-2}\ e^{-a\theta^{-1}}/t_r)} = \frac{\int_0^{\infty} \theta^{-1}e^{-a\theta^{-1}}\pi(\theta\ /t_r)d\theta}{\int_0^{\infty} \theta^{-2}e^{-a\theta^{-1}}\pi(\theta\ /t_r)d\theta} = \frac{(t_r+\mu+a)}{(r+\nu+1)}$$

So,

$\hat{\theta}_{EW} = \frac{(t_r+\mu+a)}{(r+\nu+1)}$ (3.7)

The Bayes risk of a Bayes estimator $\hat{\theta}$ of θ,corresponding to a given loss function $L(\theta,\delta)$ is given by,$B(\hat{\theta}) = E\{L(\theta,\hat{\theta})\}$.Bayes risks of Bayes estimators corresponding to four loss functions considered are in the table as follows:

3.1BAYES RISKS OF VARIOUS BAYES ESTIMATES OF θ

S.No.	Loss Function	Bayes Estimate	Bayes Risk
1.	SELF	$\hat{\theta}_B = \frac{(t_r+\mu)}{(r+\nu-1)}$	$B(\hat{\theta}_B) = \frac{(t_r+\mu)^2}{(r+\nu-1)^2(r+\nu-2)}$
2.	DLF	$\hat{\theta}_{DG} = \frac{(t_r+\mu)}{(r+\nu-2)}$	$B(\hat{\theta}_{DG}) = \frac{1}{(r+\nu-1)}$
3.	MELO	$\hat{\theta}_M = \frac{(t_r+\mu)}{(r+\nu+1)}$	$B(\hat{\theta}_M) = \frac{1}{(r+\nu+1)}$
4.	EWMELO	$\hat{\theta}_{EW} = \frac{(t_r+\mu+a)}{(r+\nu+1)}$	$B(\hat{\theta}_{EW}) = \frac{(t_r+\mu)^{r+\nu}}{(t_r+\mu+a)^{r+\nu}(r+\nu+1)}$

It is to be noted that $B(\hat{\theta}_{EW}) < B(\hat{\theta}_M) < B(\hat{\theta}_{DG})$.

5.Under the Squared Error Loss Function, the Bayes estimate of $R(t,\theta) = e^{-\frac{(pt+qt^{\alpha})}{\theta}}$, α being known, denoted by $\hat{R}_B(t,\theta)$, is given by,

$$\hat{R}_B(t,\theta) = E\{R(t,\theta)/t_r\} = \int_0^\infty R(t,\theta)\pi(\theta/t_r)d\theta = \frac{(t_r+\mu)^{r+\nu}}{\Gamma(r+\nu)}\int_0^\infty e^{-\frac{(pt+qt^2)}{\theta}}\theta^{-(\nu+r+1)}e^{-\frac{(t_r+\mu)}{\theta}}d\theta = \frac{(t_r+\mu)^{r+\nu}}{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{r+\nu}}$$

So,

$$\hat{R}_B(t,\theta) = \frac{(t_r+\mu)^{r+\nu}}{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{r+\nu}} \quad (3.8)$$

6. Under the DeGroot Loss Function the Bayes estimate of $R(t,\theta) = e^{-\frac{(pt+qt^{\alpha})}{\theta}}$, α being known, denoted by $\hat{R}_{DG}(t,\theta)$, is given by,

$$\hat{R}_{DG}(t,\theta) = \frac{E[\{R(t,\theta)\}^2/t_r]}{E\{R(t,\theta)/t_r\}} = \frac{\int_0^\infty \{R(t,\theta)\}^2\pi(\theta/t_r)d\theta}{\int_0^\infty R(t,\theta)\pi(\theta/t_r)d\theta} = \frac{\int_0^\infty e^{-\frac{2(pt+qt^2)}{\theta}}\theta^{-(\nu+r+1)}e^{-\frac{(t_r+\mu)}{\theta}}d\theta}{\int_0^\infty e^{-\frac{(pt+qt^2)}{\theta}}\theta^{-(\nu+r+1)}e^{-\frac{(t_r+\mu)}{\theta}}d\theta} = \frac{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{r+\nu}}{\{(t_r+\mu)+2(pt+qt^{\alpha})\}^{r+\nu}}$$

So,

$$\hat{R}_{DG}(t,\theta) = \frac{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{r+\nu}}{\{(t_r+\mu)+2(pt+qt^{\alpha})\}^{r+\nu}} \quad (3.9)$$

7. Under the Minimum Expected Loss (MELO) Function, the Bayes estimate of $R(t,\theta) = e^{-\frac{(pt+qt^{\alpha})}{\theta}}$, α being known, denoted by $\hat{R}_M(t,\theta)$, is given by,

$R(t,\theta) = e^{-\frac{(pt+qt^2)}{\theta}}$, denoted by $\hat{R}_M(t,\theta)$, is given by,

$$\hat{R}_M(t,\theta) = \frac{E(\theta^{-2}R(t,\theta)/t_r)}{E(\theta^{-2}/t_r)} = \frac{\int_0^\infty \theta^{-2}R(t,\theta)\pi(\theta/t_r)d\theta}{\int_0^\infty \theta^{-2}\pi(\theta/t_r)d\theta} = \frac{\int_0^\infty e^{-\frac{(pt+qt^2)}{\theta}}\theta^{-(\nu+r+3)}e^{-\frac{(t_r+\mu)}{\theta}}d\theta}{\int_0^\infty \theta^{-(\nu+r+3)}e^{-\frac{(t_r+\mu)}{\theta}}d\theta} = \frac{(t_r+\mu)^{r+\nu+2}}{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{r+\nu+2}}$$

So,

$$\hat{R}_M(t,\theta) = \frac{(t_r+\mu)^{r+\nu+2}}{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{r+\nu+2}} \quad (3.10)$$

8. Under the Exponentially Weighted Minimum Expected Loss (EWMELO) Function, the Bayes estimate of $R(t,\theta) = e^{-\frac{(pt+qt^{\alpha})}{\theta}}$, α being known, denoted by $\hat{R}_{EW}(t,\theta)$, is given by,

$$\hat{R}_{EW}(t,\theta) = \frac{E(\theta^{-2}e^{-a\theta^{-1}}R(t,\theta)/t_r)}{E(\theta^{-2}e^{-a\theta^{-1}}/t_r)} = \frac{\int_0^\infty \theta^{-2}e^{-a\theta^{-1}}R(t,\theta)\pi(\theta/t_r)d\theta}{\int_0^\infty \theta^{-2}e^{-a\theta^{-1}}\pi(\theta/t_r)d\theta} = \frac{\int_0^\infty e^{-\frac{(pt+qt^2)}{\theta}}\theta^{-(\nu+r+3)}e^{-\frac{(t_r+\mu+a)}{\theta}}d\theta}{\int_0^\infty \theta^{-(\nu+r+3)}e^{-\frac{(t_r+\mu+a)}{\theta}}d\theta} = \frac{(t_r+\mu+a)^{r+\nu+2}}{\{(t_r+\mu+a)+(pt+qt^{\alpha})\}^{r+\nu+2}}$$

So,

$$\hat{R}_{EW}(t,\theta) = \frac{(t_r+\mu+a)^{r+\nu+2}}{\{(t_r+\mu+a)+(pt+qt^{\alpha})\}^{r+\nu+2}} \quad (3.11)$$

The Bayes risk of a Bayes estimator $\hat{R}$ of $R(t,\theta)$,corresponding to a given loss function $L(\theta,\delta)$ is given by,$B(\hat{R}) = E\{L(R(t,\theta),\hat{R})\}$.Bayes risks of Bayes estimators corresponding to four loss functions considered are in the table as follows:

3.2 BAYES RISKS OF VARIOUS BAYES ESTIMATES OF $R(t,\theta)$

S.No.	Loss Function	Bayes Estimate	Bayes Risk
1.	SELF	$\hat{R}_B(t,\theta) = \frac{(t_r+\mu)^{r+\nu}}{\{(t_r+\mu)+ln(1+t^{\beta})\}^{r+\nu}}$	$B(\hat{R}_B) = \frac{(t_r+\mu)^{r+\nu}}{\{(t_r+\mu)+2(pt+qt^{\alpha})\}^{r+\nu}} - \frac{(t_r+\mu)^{2(r+\nu)}}{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{2(r+\nu)}}$
2.	DLF	$\hat{R}_{DG}(t,\theta) = \frac{\{(t_r+\mu)+ln(1+t^{\beta})\}^{r+\nu}}{\{(t_r+\mu)+2ln(1+t^{\beta})\}^{r+\nu}}$	$B(\hat{R}_{DG}) = 1 - \frac{(t_r+\mu)^{(r+\nu)}\{(t_r+\mu)+2(pt+qt^{\alpha})\}^{r+\nu}}{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{2(r+\nu)}}$
3.	MELO	$\hat{R}_M(t,\theta) = \frac{(t_r+\mu)^{r+\nu+2}}{\{(t_r+\mu)+ln(1+t^{\beta})\}^{r+\nu+2}}$	$B(\hat{R}_M) = \frac{(t_r+\mu)^{(r+\nu)}\Gamma(r+\nu+2)}{\Gamma(r+\nu)}\{\frac{1}{\{(t_r+\mu)+2(pt+qt^{\alpha})\}^{(r+\nu+2)}} - \frac{(t_r+\mu)^{(r+\nu+2)}}{\{(t_r+\mu)+(pt+qt^{\alpha})\}^{2(r+\nu+2)}}\}$
4.	EWMELO	$\hat{R}_{EW}(t,\theta) = \frac{(t_r+\mu+a)^{r+\nu+2}}{\{(t_r+\mu+a)+ln(1+t^{\beta})\}^{r+\nu+2}}$	$B(\hat{R}_{EW}) = \frac{(t_r+\mu)^{(r+\nu)}\Gamma(r+\nu++a+2)}{\Gamma(r+\nu)}\{\frac{1}{\{(t_r+\mu+a)+2(pt+qt^{\alpha})\}^{(r+\nu+2)}} - \frac{(t_r+\mu+a)^{(r+\nu+2)}}{\{(t_r+\mu+a)+(pt+qt^{\alpha})\}^{2(r+\nu+2)}}\}$

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