NUMERICAL METHODS FOR SOLVING DYNAMIC PROBLEMS

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Abstract. In the article the analysis of methods of the central difference, Newmark, Wilson and Houbolt on a concrete example is resulted. The mathematical model of a test problem is resulted. For the decision of a test problem the program of numerical calculation by means of program MATLAB 7.11.0 (R2010b) is developed and realized. Average deviations of results of calculation from results of the exact decision have made for the central difference method - 2 %, Newmark method of 3,5 %, Wilson method of 5 % and Houbolt method of 6 %. The Most exact decisions give methods of the central difference and Newmark which can be used at calculations of durability of framing.

Keywords: dynamic problems, differential equations, method integration, finite element method, direct method, method Newmark, method of the central difference, the average of the absolute deviations of points, strength of beam structures.

INTRODUCTION

The widespread use of computers in determining the strength and stiffness of structures has led to the emergence of new calculation methods, of which the finite element method, the finite difference method, and the boundary element method are the most common. When calculating beam structures using these methods, accurate results are obtained that coincide with the results of calculations by classical methods. These universal methods are also applicable for calculating large displacements when working structures outside the elastic stage and when it is dynamically loaded.

According to the method of execution and formulation of the basic equations of FEM or equations for individual finite elements, four main types of FEM are distinguished: direct, variational, residual and energy balance [1].

The direct method - similar to the deformation method in the calculation of linear supports. It is used in solving relatively simple problems, it is convenient with a clear geometric-mechanical value of the individual approximation steps. There are several varieties of the direct method. In particular, the methods of central differences, Newmark, Wilson, Houbolt.

Let's consider one of them - the Newmark method [2]. In this method, we assume that the acceleration within the Δt step remains constant (See Figure 1, a).

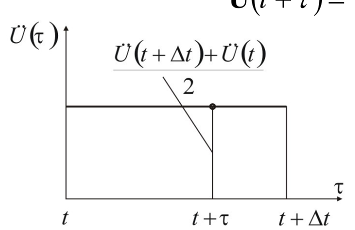
$$\ddot{\vec{\mathbf{U}}}(t+\tau)=\left[\ddot{\vec{U}}(t+\Delta t)+\ddot{\vec{U}}(t)\right]/2=const. \tag{1}$$

Integrating expression (1), we will receive

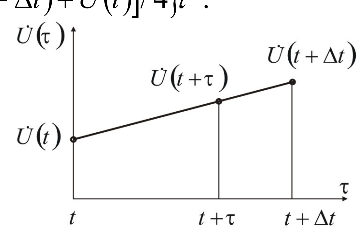
$$\dot{\vec{\mathbf{U}}}(t+\tau)=\int\left\{\left[\ddot{\vec{U}}(t+\Delta t)+\ddot{\vec{U}}(t)\right]/2\right\}d\tau=\dot{\vec{\mathbf{U}}}(t)+\left\{\left[\ddot{\vec{U}}(t+\Delta t)+\ddot{\vec{U}}(t)\right]/2\right\}\tau. \tag{2}$$

Again integrating expressions (2), we will have

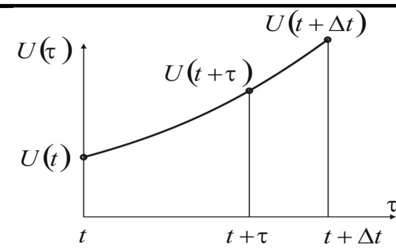
$$\vec{\mathbf{U}}(t+\tau)=\vec{\mathbf{U}}(t)+\dot{\vec{\mathbf{U}}}(t)\tau+\left\{\left[\ddot{\vec{U}}(t+\Delta t)+\ddot{\vec{U}}(t)\right]/4\right\}\tau^2. \tag{3}$$



a) linearization of acceleration in the Newmark method



b) linearization velocity in the Newmark method



c) linearization of displacement in the Newmark method

Figure 1. The scheme to step by step method

The graphs of functions (2) and (3) are shown accordingly in Figure 1, b, c. Using expressions (2) and (3), we will receive formulas for the velocity and displacement at the end of the interval Δt

$$\dot{\vec{\mathbf{U}}}(t+\Delta t)=\dot{\vec{\mathbf{U}}}(t)+\left\{\left[\ddot{\vec{U}}(t+\Delta t)+\ddot{\vec{U}}(t)\right]/2\right\}\Delta t; \qquad (4)$$

$$\vec{\mathbf{U}}(t+\Delta t)=\vec{\mathbf{U}}(t)+\dot{\vec{\mathbf{U}}}(t)\Delta t+\left\{\left[\ddot{\vec{U}}(t+\Delta t)+\ddot{\vec{U}}(t)\right]/4\right\}\Delta t^2; \qquad (5)$$

From the parity (5) we will express $\ddot{\vec{\mathbf{U}}}(t+\Delta t)$

$$\ddot{\vec{\mathbf{U}}}(t+\Delta t)=\left\{\left[4\vec{U}(t+\Delta t)-4\vec{U}(t)-4\dot{\vec{U}}(t)\Delta t\right]/\Delta t^2\right\}-\ddot{\vec{\mathbf{U}}}(t). \qquad (6)$$

Substituting (6) in (4) and resulting similar members, we will receive

$$\dot{\vec{\mathbf{U}}}(t+\Delta t)=\left[2\vec{U}(t+\Delta t)-2\vec{U}(t)-\dot{\vec{U}}(t)\Delta t\right]/\Delta t. \qquad (7)$$

Dependences (6) and (7) express acceleration $\ddot{\vec{\mathbf{U}}}(t+\Delta t)$ and velocity $\dot{\vec{\mathbf{U}}}(t+\Delta t)$ in the end of an interval Δt through displacement $\vec{\mathbf{U}}(t+\Delta t)$ in the end of the same interval (thus sizes $\vec{\mathbf{U}}(t)$, $\dot{\vec{\mathbf{U}}}(t)$, $\ddot{\vec{\mathbf{U}}}(t)$ are known from the previous step). For definition $\vec{\mathbf{U}}(t+\Delta t)$ we will work out the differential equation of motion $\mathbf{M}\ddot{\vec{\mathbf{U}}}+\mathbf{C}\dot{\vec{\mathbf{U}}}+\mathbf{K}\mathbf{U}=\vec{\mathbf{R}}$ for time moment t + Δt

$$\mathbf{M}\ddot{\vec{\mathbf{U}}}(t+\Delta t)+\mathbf{C}\dot{\vec{\mathbf{U}}}(t+\Delta t)+\mathbf{K}\vec{\mathbf{U}}(t+\Delta t)=\vec{\mathbf{R}}(t+\Delta t). \qquad (8)$$

Substituting (6) and (7) in (8), we will receive

$$\mathbf{K}_э\vec{\mathbf{U}}(t+\Delta t)=\vec{\mathbf{R}}_э, \qquad (9)$$

where $\mathbf{K}_э=\dfrac{4}{\Delta t^2}\mathbf{M}+\dfrac{2}{\Delta t}\mathbf{C}+\mathbf{K};$ (10)

$$\mathbf{R}_э(t)=\mathbf{R}(t+\Delta t)+\left(\frac{4}{\Delta t^2}\mathbf{M}+\frac{2}{\Delta t}\mathbf{C}\right)\vec{\mathbf{U}}(t)+\left(\frac{4}{\Delta t}\mathbf{M}+\mathbf{C}\right)\dot{\vec{\mathbf{U}}}(t)+\mathbf{M}\ddot{\vec{\mathbf{U}}}(t). \qquad (11)$$

Solving the equation (9), we will have

$$\vec{\mathbf{U}}(t+\Delta t)=\mathbf{K}_э^{-1}\vec{\mathbf{R}}_э(t). \qquad (12)$$

The Newmark step by step process is carried out according to formulas (10), (11) and (12). At the initial moment of time, at t_0 = 0, the displacements $\vec{\mathbf{U}}_0$ and velocities $\dot{\vec{\mathbf{U}}}_0$ of all points of the system are known, and from the differential equation of motion composed for moment t_0, we will define acceleration

$$\ddot{\vec{\mathbf{U}}}(0)=\mathbf{M}^{-1}\left[\vec{R}_0-C\dot{\vec{U}}_0-K\vec{U}_0\right]. \qquad (13)$$

Further under the formula (12) we will calculate $\vec{\mathbf{U}}(0+\Delta t)$ and under formulas (7), (6) $\dot{\vec{\mathbf{U}}}(0+\Delta t)$ and $\ddot{\vec{\mathbf{U}}}(0+\Delta t)$ etc.

Step-by-step procedure of integration by method Newmark (See Figure 2):

A. Initial calculations [3].

1. Stiffness matrices **K**, mass **M** and damping **C** are formed.
2. Initial values $\mathbf{U}_0$, $\dot{\mathbf{U}}_0$ and $\ddot{\mathbf{U}}_0$ are set.
3. The time step Δt gets out, parameters α and δ and are calculated constants:

δ≥0,50; α≥0,25(0,5+δ)^2; $a_0=\dfrac{1}{\alpha\Delta t^2}$; $a_1=\dfrac{\delta}{\alpha\Delta t}$; $a_2=\dfrac{1}{\alpha\Delta t}$; $a_3=\dfrac{1}{2\alpha}-1$;

$a_4 = \frac{\delta}{\alpha} - 1;\ a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right);\ a_6 = \Delta t(1-\delta);\ a_7 = \delta\Delta t.$

4. The effective stiffness matrix is formed $\widetilde{\mathbf{K}}$:

$$\widetilde{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}.$$

5. The matrix $\widetilde{\mathbf{K}}$ is reduced to triangular form by the Gauss method:

$$\widetilde{\mathbf{K}} = \mathbf{LDL}^T.$$

B. For each time step:

1. The effective loading for time moment is calculated t +Δt:

$$\widetilde{\mathbf{R}}_{t+\Delta t} = \mathbf{R}_{t+\Delta t} + \mathbf{M}\left(a_0\mathbf{U}_t + a_2\dot{\mathbf{U}}_t + a_3\ddot{\mathbf{U}}_t\right) + \mathbf{C}\left(a_1\mathbf{U}_t + a_4\dot{\mathbf{U}}_t + a_5\ddot{\mathbf{U}}_t\right).$$

2. There are displacements to the moment t +Δt:

$$\mathbf{LDL}^T\mathbf{U}_{t+\Delta t} = \widetilde{\mathbf{R}}_{t+\Delta t}.$$

3. Accelerations and velocities for the moment are calculated t +Δt:

$$\ddot{\mathbf{U}}_{t+\Delta t} = a_0\left(\mathbf{U}_{t+\Delta t} - \mathbf{U}_t\right) - a_2\dot{\mathbf{U}}_t - a_3\ddot{\mathbf{U}}_4;\ \dot{\mathbf{U}}_{t+\Delta t} = \dot{\mathbf{U}}_t + a_6\ddot{\mathbf{U}}_t + a_7\ddot{\mathbf{U}}_{t+\Delta t}.$$

As an example we will consider the simple system which equations of balance look like

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}.$$

Let's assume $[U]_0 = 0$ and $[\dot{U}]_0 = 0$. From the equation we will calculate $[\ddot{U}]_0$:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{U}}_0 + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix},\text{ whence } [\ddot{U}]_0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}.$$

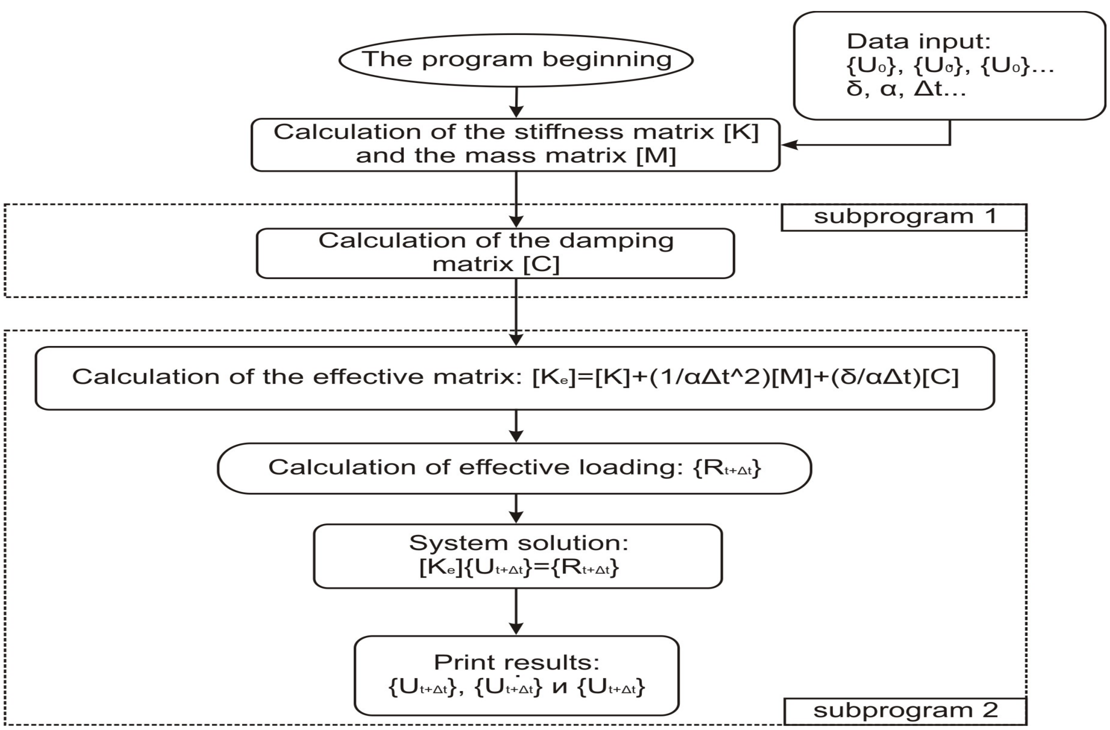


Figure 2. The block scheme

To solve the proposed equation of the mathematical model, the following numerical calculation program was developed using the MATLAB 7.11.0 (R2010b) program and implemented on computer [4].

```
K= [6 -2;-2 4]; % K - stiffness matrix (in modal coordinates)
M= [2 0; 0 1];  % M - mass matrix (in modal coordinates)
C= [0 0; 0 0];  % C - damping matrix (in modal coordinates)
dt=0.28;T=3.36;   % dt - interval
```

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U0=[0;0];       % U0 - initial displacement
U0d=[0;0];      % U0d - initial velocity
U0dd=[0;10];    % U0dd - initial acceleration
R=[0;10];       % R - force matrix (in modal coordinates)
U0dd=inv(M)*(R-C*U0d-K*U0);
% Constants used in Newmark integration
delta=0.5; alfa=1/4*(0.5+delta)^2; a0=1/(alfa*dt^2); a1= delta/(alfa*dt); a2=1/(alfa*dt);
a3=1/(2*alfa)-1; a4=(delta/alfa-1); a5=0.5*(delta/alfa-2)*dt; a6=dt*(1-delta); a7=delta*dt;
K1=K+a0*M+a1*C;
i=1;
U(:,1)=U0;Ud(:,1)=U0d;Udd(:,1)=U0dd;t=0;
fprintf('time(s)\t\tU1\t\tU2\n');
fprintf('%f\t%f\t%f\n',t,U(1,1),U(2,1));
for t=dt:dt:T
   i=i+1;    R=[0;10];
R1=R+M*(a0*U(:,i-1)+a2*Ud(:,i-1)+a3*Udd(:,i-1))+C*(a1*U(:,i-1)+a4*Ud(:,i-1)+a5*Udd(:,i-1));
  U(:,i)=inv(K1)*R1;   Udd(:,i)=a0*(U(:,i)-U(:,i-1))-a2*Ud(:,i-1)-a3*Udd(:,i-1);
  Ud(:,i)=Ud(:,i-1)+a6*Udd(:,i-1)+a7*Udd(:,i);
  fprintf('%f\t%f\t%f\n',t,U(1,i),U(2,i));
end
t=[0:dt:T];
```

Choosing time steps according to Table 1 it is possible to receive various versions of the direct method.

Table 1. Time steps to different varieties of the direct method

Central difference method	Houbolt method	Wilson method $\theta = 1{,}4$	Newmark method δ≥0,50; α≥0,25(0,5+δ)^2
$a_0 = \frac{1}{\Delta t^2};$	$a_0 = \frac{2}{\Delta t^2};$	$a_0 = \frac{6}{(\theta \Delta t)^2};$	$a_0 = \frac{1}{\alpha \Delta t^2};$
$a_1 = \frac{1}{2\Delta t};$	$a_1 = \frac{11}{6\Delta t};$	$a_1 = \frac{3}{\theta \Delta t};$	$a_1 = \frac{\delta}{\alpha \Delta t};$
$a_2 = 2a_0;$	$a_2 = \frac{5}{\Delta t^2};$	$a_2 = 2a_1;$	$a_2 = \frac{1}{\alpha \Delta t};$
$a_3 = \frac{1}{a_2}.$	$a_3 = \frac{3}{\Delta t};$	$a_3 = \frac{\theta \Delta t}{2};$	$a_3 = \frac{1}{2\alpha} - 1;$
	$a_4 = -2a_0;$	$a_4 = \frac{a_0}{\theta};$	$a_4 = \frac{\delta}{\alpha} - 1;$
	$a_5 = \frac{-a_3}{2};$	$a_5 = \frac{-a_2}{\theta};$	$a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right);$
	$a_6 = \frac{a_0}{2};$	$a_6 = 1 - \frac{3}{\theta};$	$a_6 = \Delta t(1 - \delta);$
	$a_7 = \frac{a_3}{9}.$	$a_7 = \frac{\Delta t}{2};$	$a_7 = \delta \Delta t.$
		$a_8 = \frac{\Delta t^2}{6}.$	

We have solved the above problem using the methods given in Table 1 and the results obtained are shown in Table 2.

Table 2. The results obtained

Time	**Δt**	**2Δt**	**3Δt**	**4Δt**	**5Δt**	**6Δt**
Central difference method, $\mathbf{U}_t$	0	0.0307	0.168	0.487	1.02	1.70
	0.392	1.45	2.83	4.14	5.02	5.26
Houbolt method, $\mathbf{U}_t$	0	0.0307	0.16	0.461	0.923	1.50
	0.392	1.45	2.80	4.08	5.02	5.43
Wilson method, $\mathbf{U}_t$	0.0061	0.0525	0.196	0.490	0.952	1.54
	0.366	1.34	2.64	3.92	4.88	5.31
Newmark method, $\mathbf{U}_t$	0.006733	0.050448	0.189380	0.484557	0.961314	1.580529
	0.363746	1.351041	2.683251	3.995386	4.949717	5.336621
Exact solution, $\mathbf{U}_t$	0.003	0.038	0.176	0.486	0.996	1.66
	0.382	1.41	2.78	4.09	5.00	5.29
Time	**7Δt**	**8Δt**	**9Δt**	**10Δt**	**11Δt**	**12Δt**
Central difference method, $\mathbf{U}_t$	2.40	2.91	3.07	2.77	2.04	1.02
	4.90	4.17	3.37	2.78	2.54	2.60
Houbolt method, $\mathbf{U}_t$	2.11	2.60	2.86	2.80	2.40	1.72
	5.31	4.77	4.01	3.24	2.63	2.28
Wilson method, $\mathbf{U}_t$	2.16	2.67	2.92	2.82	2.33	1.54
	5.18	4.61	3.82	3.06	2.52	2.29
Newmark method, $\mathbf{U}_t$	2.232811	2.760701	3.003509	2.850493	2.284025	1.396784
	5.129645	4.478094	3.642357	2.896744	2.435192	2.312925
Exact solution, $\mathbf{U}_t$	2.338	2.861	3.052	2.806	2.131	1.157
	4.986	4.277	3.457	2.806	2.484	2.489

In Figure 3 graphs of the results are presented in comparison with the exact solution.

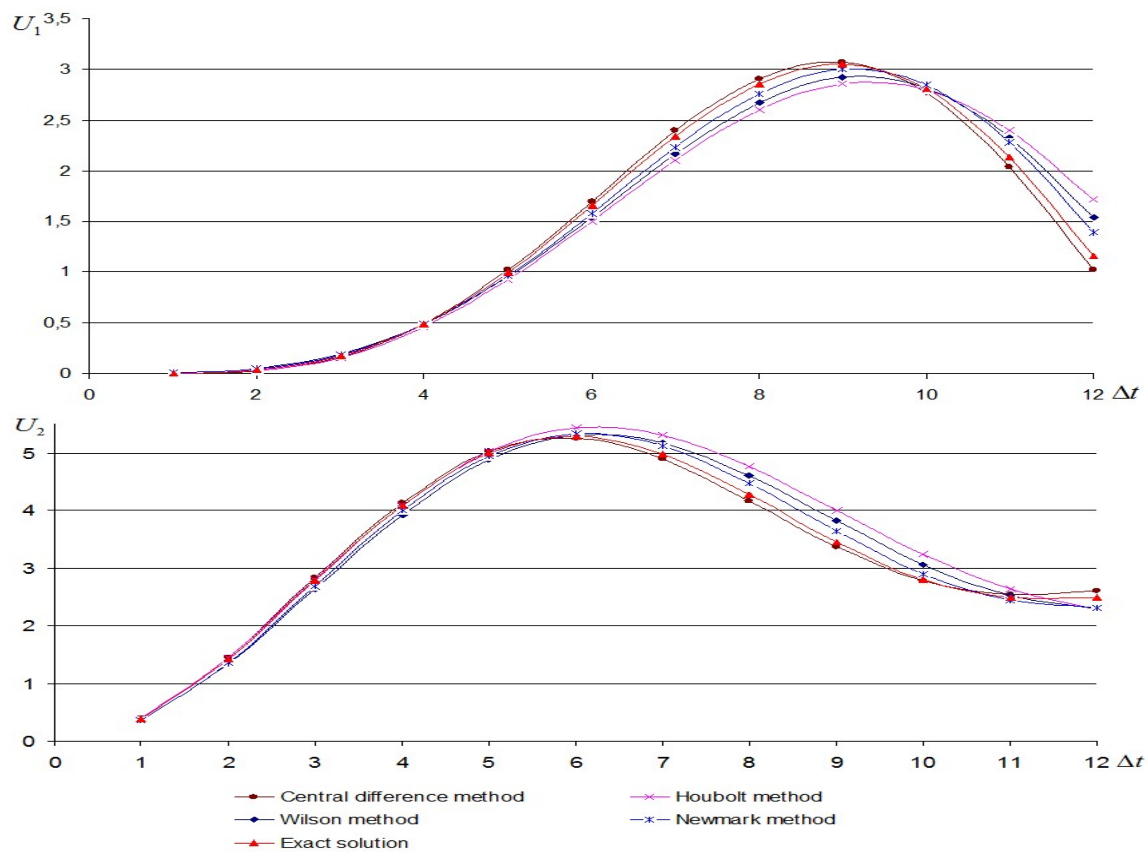

Figure 3. The displacement system

Analysis of the graphs shows that the closest results to the exact solution are the central difference method, the average of the absolute values of the deviations of the data points from the experimental one is 2%, the Newmark method is 3.5%, the Wilson method is 5% and the Houbolt method is 6%.

Thus, the analysis of direct methods of dynamic problems showed that the most accurate solutions are provided by the methods of central differences and Newmark, which can be used in the calculation of beam structures.

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