GENERAL NEW DEFINITION OF ELLIPTICAL INTEGRALS AND ITS APPLICATIONS

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Abstract
Elliptical integral has the applications in the field of the signal processing, physics and kinematics and kinetics. Limit of the integral from 0 to π/2 is the common notion in the field of mathematics but new solution to the general limit from 0 to ϕ for any arbitrary angle is presented in this paper. The novel four kind of general elliptical integral is also proposed using binomial method and the solution in terms of power series have been discussed. Computation algorithm is also discussed along with the comparison of well-known values of functions. Applications to find the perimeter of ellipse and the solution of pendulum equation show the accuracy of proposed method. Finally, the low pass elliptical filter with linear phase response is designed as illustrative example.

Keywords: Elliptical Integrals, Solution of Elliptical Integral, General Elliptical Integral, Solution of Pendulum equation, Elliptical filter, Linear phase filters

1. *Introduction*

Elliptical integrals have been begun to study from 1655 for the arc length calculation of ellipse cycloids. Newton had given the solutions of arc of ellipse in the infinite series form. The geometric problem of arc length of ellipse was given using integral form as,

$$I = \int r\left(x, \sqrt{p(x)}\right) dx \tag{1}$$

The $r(x,y)$ is the function satisfying the elliptical parameter property. $p(x)$ is the polynomial with non-repeated roots. Jacob Bernoulli in 1679 modified the elliptical integral during the calculation of arc length of spirals. Modification to equation (1) is,

$$I = \int_0^x \frac{dt}{\sqrt{(1-t^4)}} \tag{2}$$

Polar curve called as lamniscate of Bernoulli is the locus of points such that the products of distance of two fixed points are always constant [1]. Equation (3) is satisfying definition of lamniscate equation.

$$(x^2 + y^2)^2 = 2k^2 (x^2 - y^2) \tag{3}$$

Where, k is constant that satisfying the following condition,

$$k^2 = \sqrt{(x-k)^2 + y^2}\sqrt{(x+k)^2 + y^2} \tag{4}$$

And the polar form of equation (3) is,

$r^2 = k^2 cos^2\theta$ (5)

The general form of elliptical integral is,

$I = \int \frac{dx}{\sqrt{(ax^2+b)\ (cx^2+d)}}$ (6)

Definition 1: - The Incomplete Elliptical integral of the first kind is,

$F\ (\emptyset, k) = \int_0^{\emptyset} \frac{d\theta}{\sqrt{(1-K^2 \sin^2\theta)}}$ (7)

Definition 2: - The Incomplete Elliptical integral of the second kind is,

$E\ (\emptyset, k) = \int_0^{\emptyset} \sqrt{(1-K^2\sin^2\theta)}\ \ d\theta$ (8)

When, $\emptyset = \frac{\pi}{2}$; equation (7) and (8) known as complete Elliptical Integral of First and Second kind whose solution is $F\ (k)$ and $E\ (k)$ respectively.

The more general definition of the Elliptical integral is essential for the various combination of the real and complex valued functions. General new definition of Incomplete Elliptical Integral is proposed as,

Definition 3: - General New Incomplete Elliptical Integral of any kind is given as,

$I = \int_0^{\emptyset} (1+p(\theta))^r\, d\theta$ (9)

$p(\theta) = \pm K^2\sin^2\theta$ and $r = \pm\frac{1}{2}$

2. *Method of Solution*

It is assumed that the solution of equation (9) in the general infinite series form. Function $(1+p(\theta))^r$ is satisfying the condition where, $p(\theta) < 1$ and a special case when $p(\theta) = 1$

Definition 4: - Expansion of function using Binomial theorem can be generalized for $x < 1$ as,

$(1-x)^r\ \ = 1 + \sum_{n=1}^{N} (-1)^n \frac{x^n}{n!}\ \ \prod_{i=1}^{n} r-(i-1)$ (10)

$(1+x)^r\ \ = 1 + \sum_{n=1}^{N} \frac{x^n}{n!}\ \ \prod_{i=1}^{n} r-(i-1)$ (11)

Corollary 1: - The four Elliptical Integral using definition (3) and (4) for $x = \pm K^2\sin^2\theta$; $r = \pm\frac{1}{2}$; $0 \le \theta \le \emptyset$; $K \ne 1$; $K^2 \ne -1$; obtained the set of four integral equations expressed as,

$I_1 = \int_0^{\emptyset} (1-x)^r\, d\theta = \int_0^{\emptyset} (1-K^2\sin^2\theta)^{\frac{-1}{2}}\, d\theta$ (12)

$I_2 = \int_0^{\emptyset} (1-x)^r\, d\theta = \int_0^{\emptyset} (1-K^2\sin^2\theta)^{\frac{1}{2}}\, d\theta$ (13)

$I_3 = \int_0^{\emptyset} (1+x)^r\, d\theta = \int_0^{\emptyset} (1+K^2\sin^2\theta)^{\frac{-1}{2}}\, d\theta$ (14)

$I_4 = \int_0^{\emptyset} (1+x)^r\, d\theta = \int_0^{\emptyset} (1+K^2\sin^2\theta)^{\frac{1}{2}}\, d\theta$ (15)

Corollary 2: - General solution of equations (12) to (15) are expressed as,

$I_1 = \int_0^{\emptyset} (1-K^2\sin^2\theta)^{\frac{-1}{2}}\, d\theta\ = \emptyset + \sum_{n=1}^{N} (-1)^n \frac{K^{2n}}{n!}\ F\left(\frac{-1}{2}\right)_n\ G[P,K]_{2n}$ (16)

$I_2 = \int_0^{\emptyset} (1-K^2\sin^2\theta)^{\frac{1}{2}}\, d\theta\ = \emptyset + \sum_{n=1}^{N} (-1)^n \frac{K^{2n}}{n!}\ F\left(\frac{1}{2}\right)_n\ G[P,K]_{2n}$ (17)

$$I_3 = \int_0^{\emptyset}(1+K^2\sin^2\theta)^{\frac{-1}{2}}\,d\theta \;= \emptyset + \sum_{n=1}^{N} \frac{K^{2n}}{n!}\, F\left(\frac{-1}{2}\right)_n \; G[P,K]_{2n} \tag{18}$$

$$I_4 = \int_0^{\emptyset}(1+K^2\sin^2\theta)^{\frac{1}{2}}\,d\theta \;= \emptyset + \sum_{n=1}^{N} \frac{K^{2n}}{n!}\, F\left(\frac{1}{2}\right)_n \; G[P,K]_{2n} \tag{19}$$

$$F(r)_n = \prod_{i=1}^{n} r-(i-1) \tag{20}$$

$$G[P,K]_n = I_n = \int_0^{\emptyset}\sin^n\theta\,d\theta = G[P]_n + G[K]_n \tag{21}$$

$$G'[P,K]_n = I_n = \int_0^{\emptyset}\cos^n\theta\,d\theta = G'[P]_n + G'[K]_n \tag{22}$$

Remark 1: -

$$G[P]_n = \frac{-\cos\emptyset}{n}\sum_{i=1}^{m1}\sin^{n-(2*i-1)}\emptyset\,\prod_{j=1}^{m1}\frac{(n-(2*j-1))}{(n-(2*j))} \tag{23}$$

n = even and $m1 = \frac{n}{2}$

$$G[K]_n = \emptyset\,\prod_{j=1}^{m1}\frac{(n-(2*j-1))}{(n-(2*j-2))} \tag{24}$$

n = odd and $m1 = \frac{n-1}{2}$

$$G[K]_n = (1-\cos\emptyset)\,\prod_{j=1}^{m1}\frac{(n-(2*j-1))}{(n-(2*j-2))} \tag{25}$$

Remark 2: -

$$G'[P]_n = \frac{\sin\emptyset}{n}\sum_{i=1}^{m1}\cos^{n-(2*i-1)}\emptyset\,\prod_{j=1}^{m1}\frac{(n-(2*j-1))}{(n-(2*j))} \tag{26}$$

n = even and $m1 = \frac{n}{2}$

$$G'[K]_n = \emptyset\,\prod_{j=1}^{m1}\frac{(n-(2*j-1))}{(n-(2*j-2))} \tag{27}$$

n = odd and $m1 = \frac{n-1}{2}$

$$G'[K]_n = \sin\emptyset\,\prod_{j=1}^{m1}\frac{(n-(2*j-1))}{(n-(2*j-2))} \tag{28}$$

Properties: -

Consider, K=1 and $\emptyset = \frac{\pi}{2}$ and after solving equation (12),

$$I_1 = log(\sec\emptyset + \tan\emptyset) \tag{29}$$

Consider, K=1 and $\emptyset = \frac{\pi}{2}$ and after solving equation (13)

$$I_2 = \sin\emptyset \tag{30}$$

Put $\emptyset = \frac{\pi}{2}$, $I_2 = 1$

For K=0, then

$$I_1 = I_2 = I_3 = I_4 = \emptyset \tag{32}$$

Important identities

Let,

$$I_1 = K(K,\emptyset) = \int_0^{\emptyset}(1-K^2\sin^2\theta)^{\frac{-1}{2}}\,d\theta \;= \emptyset + \sum_{n=1}^{N}(-1)^n\frac{K^{2n}}{n!}\, F\left(\frac{-1}{2}\right)_n \; G[P,K]_{2n} \tag{33}$$

$$I_2 = E(K,\emptyset) = \int_0^{\emptyset}(1-K^2\sin^2\theta)^{\frac{1}{2}}\,d\theta \;= \emptyset + \sum_{n=1}^{N}(-1)^n\frac{K^{2n}}{n!}\, F\left(\frac{1}{2}\right)_n \; G[P,K]_{2n} \tag{34}$$

Then, $D(K,\emptyset) = \frac{K(K,\emptyset)-E(K,\emptyset)}{K^2} = \int_0^{\emptyset}\sin^2\theta\,(1-K^2\sin^2\theta)^{\frac{-1}{2}}\,d\theta$

$$\frac{K(K,\emptyset)-E(K,\emptyset)}{K^2} = \sum_{n=1}^{N}(-1)^n \frac{K^{2n-2}}{n!} \quad G[P,K]_{2n}\left[F\left(\frac{-1}{2}\right)_n - F\left(\frac{1}{2}\right)_n\right] \tag{35}$$

If $K' = \sqrt{(1-K^2)}$ and $\emptyset = \frac{\pi}{2}$ then $K(K') = K'(K)$, $E(K') = E'(K)$

3. *Application to find Perimeter of ellipse*

Perimeter of ellipse is given as,

$$I = 4a\int_0^{\frac{\pi}{2}}(1-e^2\cos^2\theta)^{\frac{1}{2}}d\theta = 4aE\left(e,\frac{\pi}{2}\right) \tag{36}$$

Where, a= major axis; e= eccentricity of ellipse

Case I (e=0, circle) $I = 4a\,\frac{\pi}{2} = 2\pi a$

Case II (e=1) $I = 4a$

Case III ($0 < e < 1$) $I = 4aE\left(e,\frac{\pi}{2}\right)$

4. *Application to the solution of Pendulum equation*

A simple un-damped pendulum with length "L" has motion given by equation as,

$$T(A) = 2\sqrt{\frac{2L}{g}}\int_0^A \frac{du}{\sqrt{cosu-cosA}} \tag{37}$$

Where, g= gravitational constant

u = angle between pendulum and a vertical axis

u(0)=A

T is one fourth periods, If A=0 or π there is no motion

Assume, $0 \le A \le \emptyset$; $\emptyset < \pi$

$k = sin\frac{A}{2}$, $cosA = 1-2k^2$, $sin\frac{u}{2} = ksin\theta$

$$T(A) = 4\sqrt{\frac{L}{g}}\int_0^{\emptyset}\frac{du}{\sqrt{1-K^2\sin^2\theta}}$$

$$T(A) = 4\sqrt{\frac{L}{g}}\;K(K,\emptyset) \tag{38}$$

5. *Implementation using MATLAB script*

General solution of incomplete elliptical integral is implemented using MATLAB script. The script is divided into matlab functions as,

[k1,k2]= KKK(k,phi) , [E1,E2]= EEE(k,phi)

These functions are used to evaluate solution of incomplete elliptical integral of first and second kind respectively. These functions are called by value and return the result. Desired value of k and phi is inserted which will return k1 and k2 (or E1 and E2) for first kind of elliptical integral (or second kind of elliptical integral). K1 and E1 returns the solution for the value of k whereas k2 and E2 returns the solution for the value of k' for desired value of phi.

Algorithm for [k1,k2]=KKK(k,phi) function

Initialize the variable

```
Fs1=0; Fs10=0; ter=0; n=1; a=k
 if a==1
   Fs1=log((1/cos(phi))+tan(phi));
```

```
elseif a==0
  Fs1=phi;
else
    while(ter==0)
        Fs0= ((-1)^n)*((a^(2*n))/factorial(n))* (Fun((-1/2),n))*(GPK(2*n,phi));
        Fs1=Fs1+Fs0;
        if Fs1==Fs10
            ter=1;
        else
            n=n+1;
            Fs10=Fs1;
        end
        if n>171    % for non converge value
            break;
        end
    end
    Fs1=Fs1+phi;
end
 k1=Fs1;
Fsk1=0; Fsk10=0; ter=0; n=1;
a=sqrt(1-a*a);
if a==1
  Fsk1=log((1/cos(phi))+tan(phi));
elseif a==0
  Fsk1=phi;
else
    while(ter==0)
        Fsk0= (-1)^n*((a^(2*n)/factorial(n))* Fun((-1/2),n)*GPK(2*n,phi));
        Fsk1=Fsk1+Fsk0;
        if Fsk1==Fsk10
            ter=1;
        else
            n=n+1;
            Fsk10=Fsk1;
        end
        if n>170
            break;
        end
    end
    Fsk1=Fsk1+phi;
end
k2=Fsk1;
```

6. *Important findings using present method*

Important constants are computed using the present method and found accurate using present method with minimum number of iterations (see Table 1).

Table 1Comparision of computed values

Method	Value	Actual Value	Using Present	Iterations

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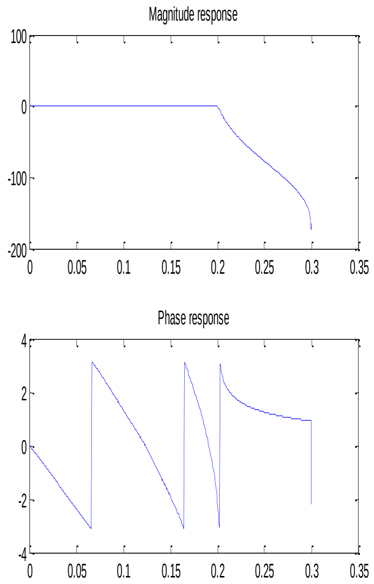
			algorithm	n required
Lemniscate const	$\frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right)$	1.311028777146060	1.311028777146060	49
Gauss Const	$\frac{\sqrt{2}}{\pi} K\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right)$	0.834626841674073	0.834626841674073	49
$K\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right)$	$\frac{\left(gamma\ \frac{1}{4}\right)^2}{4\sqrt{\pi}}$	1.854074677301372	1.854074677301372	49
$K\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\pi}{2}\right)$	$\frac{3^{\frac{1}{4}}\left(gamma\ \frac{1}{3}\right)^3}{2^{\frac{7}{3}}\ \pi}$	1.598142002112540	1.598142002112540	14
$K\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\pi}{2}\right)$	$\frac{3^{\frac{3}{4}}\left(gamma\ \frac{1}{3}\right)^3}{2^{\frac{7}{3}}\ \pi}$	2.768063145368767	2.768063145368767	171

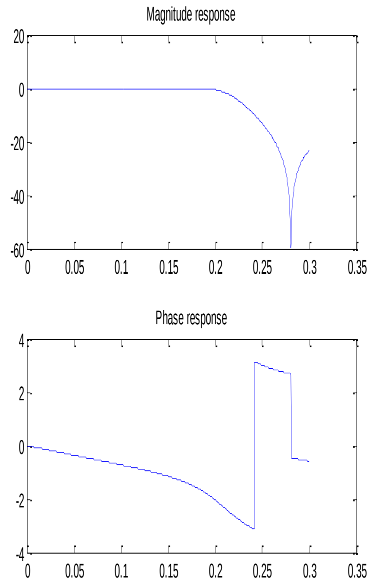
7. *Elliptical Filter Design*

Elliptical Filter is designed for the order of 12 using the traditional method and for the order 3 using the proposed method. New method requires les filter coefficients and shows the linear phase response as shown in the figure 1 and figure 2.

Input Data
Wp=0.2 * pi; % Pass band frequency
Ws = 0.3 * pi ; % Stop band frequency
Rp=0.5; % Pass band ripple
As=150; % stop band attenuation

Using Jocobian Elliptical function N=12 Using Present Method N=3





8. Conclusion

New general Elliptical integral has showed the best matching with the constants as per the table 1 that requires minimum number of iterations. Computation algorithm is implemented successfully. Applications to find the perimeter of ellipse and the solution of pendulum equation shows the accuracy of proposed method. linear phase Low pass elliptical filter is implemented.

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