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BAYESIAN ESTIMATION OF SHANNON'S ENTRPY OF THE MAXWELL'S DISTRIBUTION

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ABSTRACT

This paper deals with the Bayesian estimation of Shannon's Entropy of the Maxwell's distribution. The estimation has been done by taking Inverted Gamma distribution as the prior distribution for the unknown parameter of the distribution and four types of loss functions. A method for generating observations for the Maxwell's distribution has also been developed in order to compute and compare various estimates. Finally, various estimates have been compared with respect to their Mean and Root Mean Square Error.

Keywords: Maxwell's distribution, Bayesian Estimation, prior distribution, Weighted Squared Error Loss Function, Shannon's Entropy.

1.INTRODUCTION

Let the random variable X follows gamma distribution with parameter $\alpha = 1.5$ and $\beta = \theta$.Then the random variable $V = \sqrt{X}$ has p .d. f. given as follows:

$$f(v/\theta) = \begin{cases} 4\pi^{-05.}\theta^{-1.5}v^2e^{-\frac{v^2}{\theta}}, \text{if } v > 0, \theta > 0 \\ 0, \text{Otherwise} \end{cases} \quad (1.1)$$

The p. d. f. given by (1.1) is the p. d. f. of the Maxwell's distribution. It represents the distribution of speed (or magnitude of velocity) of a randomly chosen molecule of a gas (under certain conditions depending on pressure and flow) in a closed container. The expected value of the speed ,denoted by,μ ,is given by,

$$\mu = E(V) = 2\pi^{-0.5}\theta^{-0.5} \quad (1.2)$$

The distribution function of V,denoted by $F(v/\theta)$,is given by,

$$F(v/\theta) = \frac{\gamma(1.5, \theta^{-1}v^2)}{\Gamma(1.5)} \quad (1.3)$$

Where,

$$\gamma(p, y) = \int_0^y e^{-u}u^{p-1}du \quad (1.4)$$

represents the incomplete gamma function.

For a continuous distribution, the Shannon's Measure of Entropy (1948),denoted by, $\mathbf{S(\theta)}$,is given by,

$$S(\theta) = -E\{\ln f(V/\theta) \quad (1.5)$$

In this case,

$$S(\theta) = 0.5\ln\theta + R \quad (1.6)$$

Where,

$$R = 1.5 - \ln(4\pi^{-0.5}) - \psi(1.5) \quad (1.7)$$

Where, $\psi(.)$ is the Di-Gamma Function given by,

$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, x>0 (1.8)

It may be noted that no unbiased estimator of $\mathbf{S}(\boldsymbol{\theta})$ as given by (1.6), exists. In this paper, Bayesian estimation $\mathbf{S}(\boldsymbol{\theta})$ of has been performed under the assumption of inverted gamma distribution as the prior distribution for θ and three different forms Loss Functions.

The loss functions considered are as under:

1.The Squared Error Loss Function (SELF): In this case, the loss function denoted by $L(\theta, \delta)$, is given by,

$L(\theta, \delta) = (\theta - \delta)^2$ (1.9)

This loss function is symmetric and unbounded. It suffers from the drawback of giving equal weights to underestimation as well as to overestimation.

2.Minimum Expected Loss (MELO) Function: In this case,

$L(\theta, \delta) = \theta^{-2}(\theta - \delta)^2$ (1.10)

This loss function is asymmetric and bounded. In this case weight due to underestimation and overestimation is changed by a factor θ^{-2} as compared to the SELF. This loss function was used by Tummala and Sathe (1978) for estimating reliability of certain life time distribution and by Zellner (1979) for estimating functions of parameters in econometric models.

3. Exponentially Weighted Minimum Expected Loss (EWMELO) Function

$L(\theta, \delta) = \theta^{-2}e^{-a\theta}(\theta - \delta)^2$ (1.11)

This loss function is asymmetric and bounded. In this case weight due to underestimation and overestimation is changed by a factor $e^{-a\theta}$ as compared to the MELO and by a factor $\theta^{-2}e^{-a\theta}$ as compared to the SELF.

This type of loss function was used by the author (1997) for the first time in his work for D.Phil. SELF, MELO and EWMELO were used by Singh, the author, (1999) in the study of reliability of a multicomponent system and (2010) in Bayesian Estimation of the mean and distribution function of Maxwell's distribution. Recently, the author again used these loss functions in Bayesian estimation of function of the unknown parameter θ for the Modified Power Series Distribution (MPSD) (2021) ,for estimating Loss and Risk Functions of a continuous distribution (2021),for estimating moments and reliability of Geometric distribution. In addition to these loss functions, the author has used Degroot loss function while estimating the unknown parameter and reliability of Burr Type XII distribution (2021) and for Weibexpo distribution (2021).

A method for generating observations for the Maxwell's distribution has also been developed in order to obtain estimates of θ and the distribution function. Finally, various estimates have been compared with respect to their Mean and Root Mean Square Error.

2.BAYESIAN ESTIMATION

Let $V_1, V_2, \ldots V_n$ be a random sample of size n from the p .d . f. specified by (1.1).For observed values $v_1, v_2, \ldots v_n$, the likelihood function, ,denoted by $L(\theta)$,is given by,

$L(\theta) = k\theta^{-1.5n}e^{-\theta^{-1}t_n}$ (2.1)

Where, k is function of n,and v_i, i=1,2…n and does not contain θ.

t_n is an observed value of the statistic T_n given by,

$T_n = \sum_{i=1}^{n} V_i$ (2.2)

Since, T_n is a sufficient statistic for the family $\{f(v/\theta): \theta > 0\}$,there exists a natural conjugate family of prior densities (c.f. Raiffa and Schlaifer. ,1961).On the basis of the form of as given above, the prior density of θ is assumed to be an inverted gamma density with parameter (τ, υ)given as follows:

$$g(\theta) = \begin{cases} \frac{\tau^{\upsilon}\theta^{-(\upsilon+1)}e^{-\tau\theta^{-1}}}{\Gamma(\upsilon)}, \text{if } \theta > 0, \tau > 0, \upsilon > 0 \\ 0, \text{Otherwise} \end{cases} \quad (2.3)$$

The posterior p. d. f of θ, say ,$g_*(\theta/t_n)$,is given by,

$$g_*(\theta/t_n) = \frac{L(\theta)g(\theta)}{\int_0^\infty L(\theta)g(\theta)d\theta}$$

$$= \frac{\theta^{-(\upsilon+1.5n+1)}e^{-(t_n+\tau)\,\theta^{-1}}}{\int_0^\infty \theta^{-(\upsilon+1.5n+1)}e^{-(t_n+\tau)\,\theta^{-1}}d\theta}$$

We get, after simplification,

$$g_*(\theta/t_n) = \begin{cases} \frac{u_1^{\lambda}\theta^{-(\lambda+1)}e^{-u_1\theta^{-1}}}{\Gamma(\lambda)}, \text{if } \theta > 0 \\ 0, \text{Otherwise} \end{cases} \quad (2.4)$$

Where,

$$u_1 = t_n + \tau \quad (2.5)$$

$$\lambda = \upsilon + 1.5n \quad (2.6)$$

It is to be noted that the posterior p. d. f. is also inverted gamma distribution with parameters $u_1$and λ .

1.Under the Squared Error Loss Function given by, $L(S(\theta), \delta) = (S(\theta) - \delta)^2$, the Bayes estimate of $S(\theta)$,denoted by $\hat{S}_B$, is given by,

$$\hat{S}_B = E\{S(\theta)\,/t_n\} = \int_0^\infty S(\theta)g_*(\theta/t_n)d\theta = \frac{\ln u_1 - \psi(\lambda)}{2} + R$$

So,

$$\hat{S}_B = \frac{\ln u_1 - \psi(\lambda)}{2} + R \quad (2.4)$$

2.Under the Minimum Expected Loss (MELO) Function, given by, $L(S(\theta), \delta) = \theta^{-2}(S(\theta) - \delta)^2$, the Bayes estimate of $S(\theta)$,(also known as the Minimum Expected Loss (MELO) Estimate, denoted by $\hat{S}_M$, is given by,

$$\hat{S}_M = \frac{E(\theta^{-2}S(\theta)\,/t_n)}{E(\theta^{-2}\,/t_n)} = \frac{\int_0^\infty \theta^{-2}S(\theta)g_*(\theta/t_n)d\theta}{\int_0^\infty \theta^{-2}g_*(\theta/t_n)d\theta} = \frac{\ln u_1 - \psi(\lambda_1)}{2} + R$$

So,

$$\hat{S}_M = \frac{\ln u_1 - \psi(\lambda_1)}{2} + R \quad (2.5)$$

Where, $\lambda_1 = \lambda + 2$

3.Under the Exponentially Weighted Minimum Expected Loss (EWMELO) Function, given by, $L(S(\theta), \delta) = \theta^{-2}e^{-a\theta^{-1}}(S(\theta) - \delta)^2$, the Bayes estimate of $S(\theta)$,known as the Exponentially Weighted Minimum Expected Loss (MELO) Estimate, denoted by $\hat{S}_{EW}$, is given by,

$$\hat{S}_{EW} = \frac{E(\theta^{-2}e^{-a\theta^{-1}}S(\theta)\,/t_n)}{E(\theta^{-2}e^{-a\theta^{-1}}\,/t_n)} = \frac{\int_0^\infty \theta^{-2}e^{-a\theta^{-1}}S(\theta)g_*(\theta/t_n)d\theta}{\int_0^\infty \theta^{-2}e^{-a\theta^{-1}}g_*(\theta/t_n)d\theta} = \frac{\ln(u_1+a) - \psi(\lambda_1)}{2} + R$$

So,

$$\hat{S}_{EW} = \frac{\ln(u_1+a) - \psi(\lambda_1)}{2} + R \quad (2.6)$$

The Maximum Likelihood Estimator (M.L.E) of $S(\theta)$,denoted by $\hat{S}_1$,is given by,

$$\hat{S}_1 = \frac{\ln(\frac{2t_n}{3n})}{2} + R \quad (2.7)$$

Taking n=10, θ=4 ,10 sets of observations have been simulated for the Maxwell's distribution. Bayes Estimates of S(θ),under various forms of loss functions as given above have been computed and compared with respect to their Mean and R.M.S.E. The actual value of S(θ) is 1.34273.

It has been observed that EWMELO estimates have smallest R.M.S.E of all other estimates.

Results obtained have been shown in the table as under:

2.1 VARIOUS ESTIMATES AND THEIR COMPARISON

S.No.	t_n	$\hat{S}_1$	$\hat{S}_B$	$\hat{S}_M$	$\hat{S}_{EW}$,a=6	$\hat{S}_{EW}$,a=8
1.		1.47379	1.43872	1.38153	1.4177	1.4292
2.	66.3286	1.39287	1.35999	1.3028	1.34488	1.35816
3.	71.0767	1.42744	1.39358	1.33639	1.37585	1.38833
4.	66.6796	1.39551	1.36255	1.30536	1.34724	1.36045
5.	50.2876	1.25443	1.22621	1.16902	1.22333	1.2402
6.	40.8922	1.15102	1.12717	1.06998	1.13544	1.15549
7.	61.0950	1.35177	1.32015	1.26296	1.30838	1.33264
8.	59.9597	1.34239	1.31107	1.25388	1.30009	1.3146
9.	83.5411	1.50822	1.47232	1.41513	1.44903	1.45984
10.	68.4728	1.40877	1.37544	1.31825	1.3591	1.37201
MEAN		1.37062	1.33872	1.28153	1.3261	1.34009
R.M.S.E.		0.10297	0.101046	0.118067	0.093249	0.089094

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