CHARACTERIZATION OF INDUCED ORDER STATISTICS FROM RAYLIEGH BIVARIATE DISTRIBUTION

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ABSTRACT
In this paper, a new bivariate distribution, using the concept given by Mongestern (1956), has been derived from univariate Rayleigh distribution. This new bivariate Rayleigh distribution is then considered to obtain the Induced (concomitants) order statistics. We have finally derived the probability density function of a single and joint induced order statistics and have obtained their moments to characterize the distribution.

Keywords: Induced Order Statistics, Rayliegh Bivariate Distribution.

1. INTRODUCTION

In probability theory and statistics, the Rayleigh distribution is a continuous probability distribution. A Rayleigh distribution is often observed when the overall magnitude of a vector is related to its directional components. The example of this feature is that the Rayleigh distribution naturally arises when wind speed is analyzed into its orthogonal 2-dimensional vector components. Assuming the magnitude of each component is uncorrelated and normally distributed with equal variance then the overall wind speed (vector magnitude) will be characterized by a Rayleigh distribution. The distribution can also arise in the case of random complex numbers whose real and imaginary components are iid Gaussian. In that case, the absolute value of the complex number is Rayleigh-distributed. The distribution is named after Lord Rayleigh. Chi-square, Rice distribution and Weibull distribution are the generalization of the Rayleigh distribution.
In this present paper, a new bivariate distribution has been derived from univariate Rayleigh distribution using the concept given by Mongestern (1956). This bivariate Rayleigh distribution is then considered to obtain the Induced (concomitants) order statistics.
In the section 2, characterization of univariate Rayliegh distribution and bivariate Rayliegh distribution has been developed. In section 3 and 4 the density function of order statistics and induced order statistics have been obtained. To characterize the model, the moments of induced order statistics have been also obtained. The moment generating functions and cumulant generating function has been obtained. The joint distribution of two concomitants is also obtained in section 5.

2 CHARACTERISTICS OF RAYLEIGH DISTRIBUTION AND BIVARIATE RAYLEIGH DISTRIBUTION

In case the absolute value of the complex number is Rayleigh–distributed. The distribution is name after Lord Rayleigh. The p.d.f of Rayleigh distribution is given by

$$f(x) = \frac{xe^{-x^2/2\sigma^2}}{\sigma^2} \qquad 0 \le x \le \infty \qquad \dots(2.1)$$

The cumulative distribution function of above distribution calculates to

$$F(x) = \int_o^x f(x)dx \qquad 0 \le x \le \infty$$

$$= 1 - e^{-x^2/2\sigma^2}$$

The reliability function at time t say R (T) = P (T>t) is given by

$$R(X) = p(X > x) = e^{-x^2/2\sigma 2}$$

Characteristics of the Rayleigh Distribution
Mean : The arithmetic mean for the Rayleigh distribution is:

$$E(X) = \int_0^\infty xf(x)dx = \sigma\sqrt{\frac{\pi}{2}}$$

Variance: The variance for the Rayleigh distribution is:

$$V(X) = E(X^2) - (E(X))^2 = \frac{4-\pi}{2}\sigma^2$$

Median: The median of the Rayleigh distribution can be obtained as:
By the definition of median, we have

$$\int_0^{M_e} x/\sigma^2 \, e^{-x^2/2\sigma^2} \, dx = 1/2$$

$$Me = \sigma\sqrt{\log(4)}$$

Bivariate Rayleigh distribution:
A bivariate distribution has been developed by using the concept of Mongestern (1956) as follows:
Putting X and Y follows univariate Rayleigh distribution with p.d.f. defined in eq. (2.1)
Now, to derive a bivarite distribution the following form is used:-

$$F(x,y) = F(x)F(y)\left[1+\delta(1-F(x))\right]\left[1-F(y)\right] \qquad ...(2.2)$$

For δ=-1

$$= 1 - e^{-x^2/2\sigma^2} - e^{-y^2/2\sigma^2} + e^{\frac{-x^2-2y^2}{2\sigma^2}} + e^{\frac{-2x^2-y^2}{2\sigma^2}} - e^{\frac{-x^2-y^2}{\sigma^2}} \qquad \ldots(2.3)$$

Probability density function:

The p.d.f. of the bivariate Rayleigh distribution will be

$$f(x,y) = \frac{d^2}{dx\,dy}\left[f(x,y)\right] = \frac{2xy}{\sigma^4}\left[e^{\frac{-x^2-2y^2}{2\sigma^2}} + e^{\frac{-2x^2-y^2}{2\sigma^2}} - 2e^{\frac{-x^2-y^2}{\sigma^2}}\right] \qquad \ldots(2.4)$$

The conditional pdf of x given y is

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{2x}{\sigma^2}\left[e^{\frac{-x^2-y^2}{2\sigma^2}} + e^{\frac{-2x^2}{2\sigma^2}} - 2e^{\frac{-2x^2-y^2}{2\sigma^2}}\right] \qquad \ldots(2.5)$$

Conditional pdf of y given x is

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{2y}{\sigma^2}\left[e^{\frac{-2y^2}{2\sigma^2}} + e^{\frac{-x^2-y^2}{2\sigma^2}} - 2e^{\frac{-x^2-2y^2}{2\sigma^2}}\right] \qquad \ldots(2.6)$$

3. PROBABILITY DENSITY FUNCTION OF INDUCED ORDER STATISTICS

In this section, density function of Order Statistics and their Induced (Concomitants) Order Statistics have been obtained. For Rayleigh distribution the pdf of the r^{th} order statistics can be obtained as:

$$f_{r:n} = \frac{n!}{(r-1)!(n-r)!}\left[F(x)^{r-1}(1-F(x)\right]^{n-r} f(x)$$

$$= C_{r:n}\,\frac{x}{\sigma^2}\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{r-1}\left(e^{\frac{-x^2}{2\sigma^2}}\right)^{n-r+1} \qquad \ldots(3.1)$$

For r=1 the pdf of the first order statistics

$$= C_{1:n}\,\frac{x}{\sigma^2}\left(e^{\frac{-x^2}{2\sigma^2}}\right)^{n} \qquad ...(3.2)$$

For r=n the probability density function of the n^{th} order statistics

$$= C_{n:n}\,\frac{x}{\sigma^2}\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{n-1} e^{\frac{-x^2}{2\sigma^2}} \qquad \ldots(3.3)$$

The probability density function of the n^{th} concomitant of the order statistic for r=n is

$$g_{(n:n)}(y)=\int f(y/x)\, f_{n:n}(x)\,dx=\frac{2y}{\sigma^4}C_{n:n}\left[\int_0^\infty x\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{n-1} e^{\frac{-x^2-2y^2}{2\sigma^2}}dx+\right.$$

$$\left.\int_0^\infty x\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{n-1} e^{\frac{-2x^2-y^2}{2\sigma^2}}dx-\int_0^\infty 2x\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{n-1} e^{\frac{-2x^2-2y^2}{2\sigma^2}}\right]dx$$

$$g_{(n:n)}(y)==\frac{2y}{\sigma^2}e^{\frac{-y^2}{\sigma^2}}+\frac{2yn}{\sigma^2 B(2,n)}e^{\frac{-y^2}{2\sigma^2}}-\frac{4ny}{\sigma^2 B(2,n)}e^{\frac{-y^2}{\sigma^2}} \qquad \ldots(3.4)$$

The probability density function of $y_{(r:n)}$ i.e. r[th] Induced order statistic will be

$$g_{(r:n)}(y)=\sum_{i=r}^{n}(-1)^{i-r}\, i-1_{C_r}\; n_{C_i}\; g_{(i:i)}(y)$$

$$=\sum_{i=r}^{n}(-1)^{i-r}\, i-1_{C_r}\; n_{C_i}\left[\frac{2y}{\sigma^2}e^{-y^2/\sigma^2}+\frac{2yi}{\sigma^2B(2,i)}e^{-y^2/2\sigma^2}-\frac{4i}{\sigma^2B(2,i)}\,y\,e^{-y^2/\sigma^2}\right] \ldots(3.5)$$

We can obtain the probability density function of first concomitant of the order statistics by putting r=1 we have:

$$g_{(1:n)}(y)=\sum_{i=r}^{n}(-1)^{i-r}\, i-1_{C_r}\left[\frac{2y}{\sigma^2}e^{-y^2/\sigma^2}+\frac{2y}{\sigma^2B(2,1)}e^{-y^2/2\sigma^2}-\frac{4}{\sigma^2B(2,1)}\,y\,e^{-y^2/\sigma^2}\right] \ldots(3.6)$$

Moments of $y_{(n:n)}$:

The k[th] moment of $y_{(n:n)}$ k=0,1,2,…….n.

$$\mu^k_{(n:n)}=E\left[y^k_{(n:n)}\right]=\int_0^\infty y^k\, g_{(n:n)}(y)\,dy=\sigma^k\,\overline{|k/2+1}\left[1+\frac{2n\sigma^k 2^{k/2+1}}{B(2,n)}-\frac{2n}{B(2,n)}\right] \quad ..(3.7)$$

Similarly one can obtain the expression for k[th] moments of $y_{(r:n)}$ as:

$$\mu^k_{(r:n)}=E\left[y^k_{(n:n)}\right]=\int_0^\infty y^k\, g_{(n:n)}(y)\,dy$$

$$=\sum_{i=r}^{i}(-1)^{i-r}\, i-1_{C_{r-1}}\left[\sigma^k\,\overline{|k/2+1}\left[1+\frac{i\,2^{k/2+1}}{B(2,i)}-\frac{2i}{B(2,i)}\right]\right. \qquad ...(3.8)$$

4 MOMENT GENERATING FUNCTION AND CUMULANT GENERATING FUNCTION OF $Y_{[n:n]}$:

Moment generating function of $y_{(n:n)}$ is given by:

$$M_{(n:n)}(t)=E\left[e^{ty}\,g_{(n:n)}\right]$$

$$=\int_0^\infty \frac{2}{\sigma^2}e^{ty}\,y\,e^{-y^2/\sigma^2}dy+\int_0^\infty\frac{2n}{\sigma^2B(2,n)}e^{ty}\,y\,e^{-y^2/2\sigma^2}dy-$$

$$\frac{4n}{\sigma^2B(2,n)}\int_0^\infty e^{ty}\,y\,e^{-y^2/\sigma^2}dy$$

$$M_{(n:n)}(t)=\frac{2\sqrt{z}\,e^{\sigma t\sqrt{z}-z}}{\sigma t-2\sqrt{z}}+\frac{2n}{B(2,n)}\frac{2\sqrt{z}\,e^{\sigma t\sqrt{z}-z}}{\sigma t-\sqrt{2z}}-\frac{2n}{B(2,n)}\frac{2\sqrt{z}\,e^{\sigma t\sqrt{z}-z}}{\sigma t-2\sqrt{z}} \qquad \ldots(4.1)$$

Now to obtain the moments by expansion method, one can use the following expression.

$$\mu^1_r=Coefficient\ of\ \frac{t^r}{\underline{|r}}$$

Differentiating it in this manner upto k times, we get the k[th] moment of $y_{(r:n)}$:

$$\frac{d^k}{dt^k}\,\mu_{(r:n)}\,(t)=\frac{2\sqrt{z}\;e^{\sigma t\sqrt{z}-z}}{\sigma\,t-2\sqrt{z}}+\frac{2n}{B(2,n)}\,\frac{\sqrt{2z}\;e^{\sigma t\sqrt{2z}-z}}{\sigma\,t-\sqrt{2z}}-\frac{2n}{B(2,n)}\,\frac{2\sqrt{z}\;e^{\sigma t\sqrt{2z}-z}}{\sigma\,t-2\sqrt{z}}$$

$$\mu_1=\;\frac{2\sqrt{z}\;e^{\sigma t\sqrt{z}-z}}{\sigma\,t-2\sqrt{z}}+\frac{2\sqrt{2z}\;e^{\sigma t\sqrt{2z}-z}}{B(2,1)\sigma\,t-\sqrt{2z}}-\frac{2.2\sqrt{z}\;e^{\sigma t 2\sqrt{z}-z}}{B(2,1)\sigma\,t-2\sqrt{z}}$$

$$\mu_2=\;\frac{2\sqrt{z}\;e^{\sigma t\sqrt{z}-z}}{\sigma\,t-2\sqrt{z}}+\frac{4\sqrt{2z}\;e^{\sigma t\sqrt{2z}-z}}{B(2,2)\sigma\,t-\sqrt{2z}}-\frac{4\sqrt{z}\;e^{\sigma t 2\sqrt{z}-z}}{B(2,2)\sigma\,t-2\sqrt{z}}$$

5. JOINT DISTRIBUTION OF TWO CONCOMITANTS $Y_{(r:n)}$ AND $Y_{(s:n)}$:

The joint p.d.f of $Y_{(r:n)}$ and $Y_{(s:n)}$ is given by

$$g_{(r:s:n)}\,(y_1,y_2)=\int_0^\infty\int_0^{x_2} f(y_1/x_1)\,f(y_2/x_2)\,f_{r:s:n}(x_1\,x_2)\,dx_1\,dx_2 \qquad \ldots.(5.1)$$

The joint p.d.f of $X_{r:n}$ and $X_{s:n}$ for the bivariate Rayleigh distribution with pdf

$$f_{(r:s:n)}(x_{1,}x_2)=C_{r:s:n}\left[F(x_1)\right]^{r-1}\left[F(x_2)-F(x_1)\right]^{s-r-1}\left[1-f(x_2)\right]^{n-s}\;f(x_1)\,f(x_2)\;\ldots.(5.2)$$

$$C_{r,s:n}=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

$$=C_{r:s:n}\sum_{j=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{r-1-j}\,(-1)^{s-r-1-k}\;r-1_{C_j}\;s-r-1_{C_k}\left[\exp\left(\frac{-x_1^2}{2\sigma^2}\right)\right]^{r-j+k}\left[\exp\left(\frac{-x_2^2}{2\sigma^2}\right)\right]^{n-r-k}\frac{x_1x_2}{\sigma^4}\qquad\ldots(5.3)$$

Putting the value eq.(2.18) in eq.(2.16)

$$g_{(r:s:n)}\,(y_1,y_2)=\int_0^\infty\int_0^{x_2} f\left(\frac{y_1}{x_1}\right)f\left(\frac{y_2}{x_2}\right)f_{r:s:n}(x_1\,x_2)\,dx_1\,dx_2$$

$$g_{(r,s:n)}\,(y_1,y_2)=\frac{4y_1y_2\,\sigma^4}{\sigma^8}\;C_{r:s:n}\sum_{j=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{r-1-j}\,(-1)^{s-r-1-k}\;r-1_{C_j}\,s-r-1_{C_k}$$

$$\left[\exp\left(\frac{-y_1^2}{\sigma^2}\right)\frac{\exp\left(-(r-j+k)\,\frac{x_2^2}{2\sigma^2}-1\right.}{-(r-j+k)}\right]+$$

$$\exp\left(\frac{-y_1^2}{2\sigma^2}\right)\left[\frac{\exp\left(-(r-j+k+1)\,\frac{x_2^2}{2\sigma^2}-1\right.}{-(r-j+k+1)}\right]-2\exp\left(\frac{-y_1^2}{\sigma^2}\right)$$

$$\left[\frac{\exp\left(-(r-j+k+1)\,\frac{x_2^2}{2\sigma^2}-1\right.}{-(r-j+k+1)}\right]\times$$

$$\left[-\frac{\exp\left(\frac{-y_2^2}{\sigma^2}\right)}{(n-r-k)}-\frac{\exp\left(\frac{-y_2^2}{2\sigma^2}\right)}{(n-r-k+1)}+\frac{2\exp\left(\frac{-y_2^2}{\sigma^2}\right)}{(n-r-k+1)}\right]$$

$$=\frac{4y_1\,y_2}{\sigma^4}\,C_{r:s:n}\sum_{j=0}^{r-1}\sum_{k=0}^{s-r-1}(-1)^{r-1-j}\,(-1)^{s-r-1-k}\;r-1_{C_j}\,s-r-1_{C_k}$$

$$\left[\exp\left(\frac{-y_1^2}{\sigma^2}\right)\left(\frac{\exp\left(-\left(r-j+k\right)\frac{x_2^2}{2\sigma^2}-1\right.}{-\left(r-j+k\right)}\right)+\exp\left(\frac{-y_1^2}{2\sigma^2}\right)\right.$$

$$\frac{\exp\left(-\left(r-j+k+1\right)\frac{x_2^2}{2\sigma^2}-1\right.}{-\left(r-j+k+1\right)}-2\exp\left(\frac{-y_1^2}{\sigma^2}\right)$$

$$\left[\frac{\exp\left(-\left(r-j+k+1\right)\frac{x_2^2}{2\sigma^2}-1\right.}{-\left(r-j+k+1\right)}\right]\left[-\frac{\exp\left(\frac{-y_2^2}{\sigma^2}\right)}{\left(n-r-k\right)}-\right.$$

$$\left.\frac{\exp\left(\frac{-y_2^2}{2\sigma^2}\right)}{\left(n-r-k+1\right)}+\frac{2\exp\left(\frac{-y_2^2}{\sigma^2}\right)}{\left(n-r-k+1\right)}\right] \quad \ldots(5.4)$$

2.6. CONCLUSION

We have developed a new bivariate distribution named as Rayleigh Bivariate distribution by using concept of Mongestern (1956). The distribution of order statistics and their induced statistics has been obtained. Further, the moments of concomitants of order statistics have been also obtained that characterize the distribution.

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